NECESSARY AND SUFFICIENT CONDITIONS OF OPEN-LOOP AND CLOSED-LOOP SOLVABILITY FOR DELAYED STOCHASTIC LQ OPTIMAL CONTROL PROBLEMS[*]

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Abstract. In this paper, a linear quadratic optimal control problem driven by a stochastic differential delay system is investigated, where both state delay and control delay can appear in the state equation, especially in the diffusion term. Three kinds of solvability for the delayed control problem are proposed: the open-loop solvability, the closed-loop representation of open-loop optimal control, the closed-loop solvability, and their necessary and sufficient conditions are obtained. The delayed control problem is transformed into an infinite dimensional optimal control problem without delay but with a new control operator. Some novel auxiliary equations are constructed to overcome the difficulties caused by the new control operator, because state delay and control delay coexist, and some stochastic analysis tools are lacking in the study of the above three kinds of solvability. The open-loop solvability is assured by the solvability of a constrained forward-backward stochastic evolution system and a convexity condition, or by the solvability of an anticipated-backward stochastic differential delay system and a convexity condition; the closed-loop representation of the open-loop optimal control is given via a coupled matrix-valued Riccati equation; the closed-loop solvability is assured by the solvability of an operator-valued Riccati equation or a coupled matrix-valued Riccati equation.

Key words. linear quadratic control, time delay, open-loop solvability, closed-loop solvability, Riccati equation

AMS subject classifications. 93C25, 49K15, 49K27, 49N10

1. Introduction.

Many problems can be regarded as optimal control problems in the fields of economy, finance, aerospace, network communication and so on (see [3, 5, 7]). In the real world, the development of certain phenomena depends not only on the present state, but also on the past state trajectories. After a controller exerts control, it takes some time to have a practical effect on the control systems. Meanwhile, the development of control systems is affected by some uncertainties. Therefore, how to obtain the optimal control of stochastic control systems with both state delay and control delay, has become the core problem of control theory.

Delayed control systems have wide background and applications (see [3,7,9,13,14, 24,26]). For example, we consider a pension fund model introduced in [7], and modify it to take into account the time of implementing the portfolio strategy. Suppose that the manager can invest in two assets: a risky asset (e.g. stock) and a riskless asset

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(e.g. bond). Then, the wealth equation is as follows:

$$\begin{cases} dx(t)=\big[rx(t)+\sigma\lambda u(t-\delta)\big]dt-\big[q+f(x(t)-x(t-\delta))\big]dt+\sigma u(t-\delta)dW(t), 0\leqslant t\leqslant T,\\ x(\theta)=\varphi(\theta), \quad u(\theta)=\psi(\theta),\ \theta\in[-\delta,0], \end{cases}$$

where $x(\cdot)$ is the fund wealth, $u(\cdot)$ is the amount of money invested in the risky asset, $r\geqslant 0$ is the instantaneous return rate of the riskless asset, $\mu\geqslant r$ is the instantaneous rate of expected return of the risky asset and $\sigma>0$ is the instantaneous rate of volatility. Assume that μ can be expressed by the relation $\mu=r+\sigma\lambda$, where $\lambda\geqslant 0$ is the instantaneous risk premium of the market. Compared with the classical self-financing portfolio model, $u(t-\delta)$ considers the time of implementing the portfolio strategy, and the difference $q+f(x(t)-x(t-\delta))$ represents the external cashflows of contributions and benefits which enter the dynamics of the fund. The portfolio strategy $u(t-\delta)$ at time $t-\delta$ is executed at time t, when the asset prices and the fund wealth have already changed. q is the difference between the exiting cashflow of the aggregate benefits, paid by the fund as a minimum guarantee to its members in retirement, and the entering cashflow, paid by the members who are adhering to the fund. f is a constant, and the term $f(x(t)-x(t-\delta))$ represents the dividends to members when the investment is profitable or the replenishment of cash flow when the investment is loss-making. $\varphi(\cdot)$ is the initial wealth or the fund donated at $[-\delta,0]$, and $\psi(\cdot)$ is the initial investment strategy according to $\varphi(\cdot)$. The manager wants to achieve the expected return a, that is, he would like to minimize the following objective functional:

$$J(\varphi(\cdot),\psi(\cdot);u(\cdot))=\mathbb{E}\big[|x(T)-a|^2\big].$$

In the above, we use a single delay to describe the time of implementing the portfolio strategy. In fact, in the fields such as biology, physics and medicine, a single delay cannot adequately describe the dynamics of a system, multiple pointwise delays and distributed delay have to be used (see [13, 14, 24]), because the time required for plants and animals to grow and mature varies significantly, the transport and diffusion rates of substances are also different, and sometimes these delay effects show smooth changes in time, rather than instantaneous responses.

Motivated by these practical examples, we would like to study stochastic linear quadratic optimal control problems with both state delay and control delay. In the 18th century, Euler, Bernoulli, Lagrange, Laplace and Poisson firstly considered delay systems when studying various geometric problems. For deterministic delayed optimal control problems, Delfour in [6] solved a linear quadratic optimal control problem with pointwise and distributed state delay by the product space approach. Later, Vinter and Kwong in [32] reformulated a linear differential delay system with distributed control delay as an evolution system with bounded control operators by the structural state method. Ichikawa in [12] studied an optimal control problem with pointwise control delay by the extended state method. Subsequently, massive research results have been produced, such as [1,2]. *Stochastic differential delay equations* (SDDEs) are usually used to describe the dynamics of delayed stochastic systems, more references can be referred to [25,26]. So far, optimal control problems of stochastic differential delay systems have been extensively studied. When only state delay appears in control systems, Flandoli in [8] transformed the delayed optimal control problem into an abstract one in Hilbert space, then derived the optimal feedback. Liang et al. in [19] applied the method of completion of squares to obtain the feedback of the optimal control. When only control delay appears in control systems, Wang and Zhang in [33] described equivalently the stochastic control systems with input delay by an abstract model without delay in a Hilbert space, then derived the feedback of the optimal control. Zhang and Xu in [36] gave the solvability condition of the optimal control and the analytical controller based on a modified Riccati differential equation. For more literature, readers can be referred to [7, 11, 23] (for stochastic optimal control problems with state delay only) and [3,11,34] (for stochastic optimal control problems with control delay only). However, when state delay and control delay both appear in control systems, most literature only studied the maximum principle for the optimal control, and did not provide the feedback of the optimal control (see [5, 9, 17, 35]).

Recently, Sun and Yong in [29] firstly found that there is a significant difference between open-loop and closed-loop saddle points for a stochastic linear quadratic two-person zero-sum differential game. As a continuation work of [29], Sun et al. in [28] studied the open-loop and closed-loop solvability for stochastic linear quadratic optimal control problems, and established the equivalence between the strongly regular solvability of the Riccati equation and the uniform convexity of the cost functional. Ni et al. in [27] considered a stochastic linear quadratic problem with transmission delay, and characterized its solvability by Riccati-like equations and linear matrix equality-inequalities. As for related problems in an infinite time horizon, Sun and Yong in [30] discussed a stochastic linear quadratic optimal control problem with constant coefficients and researched the open-loop and closed-loop solvability. Li et al. in [18] presented a systematic theory for two-person non-zero sum differential games of mean-field type stochastic differential systems with quadratic performance in an infinite time horizon. In the aspect of infinite dimensional problems, Lü generalized [28] to a stochastic linear quadratic optimal control problem governed by a stochastic evolution system in [20], and put two strict assumptions. Later Lü in [21] dropped them, gave the closed-loop solvability for a linear quadratic optimal control problem driven by a mean-field type stochastic evolution system, and improved the main results in [20] noticeably.

This paper investigates a stochastic linear quadratic optimal control problem involving both state delay and control delay, the optimal control consists of three parts at least: the first one is proportional to the current value of the state, the second one involves an integral of the state trajectory over the past time interval, and the third one involves an integral of the control trajectory over the past time interval. The structure of the optimal control is so complex, therefore, how to define the closed-loop solvability for the delayed stochastic optimal control problem? After the appropriate definitions are introduced, how to characterize the closed-loop solvability?

The contributions and innovations in this paper are summarized as follows:

- A very general model is studied. Both state delay and control delay can appear in the state equation and the cost functional, especially in the diffusion term. When the original delayed system is transformed into an infinite dimensional control system without delay, the new control operators appear and can not be dealt with using the existing methods (see [8, 15, 16, 19, 33, 36]). Thus, some new approaches are constructed to overcome the above difficulties.
- Three kinds of solvability are proposed: the open-loop solvability, the closed-loop representation of the open-loop optimal control and the closed-loop solvability for the original delayed stochastic optimal control problem. To characterize them, an equivalent optimal control problem without delay is constructed, and then the open-loop and closed-loop solvability are defined.
- Some necessary and sufficient conditions for the above three kinds of solvability are derived.

(a) The open-loop solvability is assured by the solvability of a constrained forward-backward stochastic evolution system and a convexity condition. A novel backward equation is introduced as an adjoint equation, since the new control operators make the transformed problem not a standard infinite dimensional stochastic optimal control problem, and its existence and uniqueness is proved by an equivalent backward stochastic evolution equation. Moreover, a clearer equivalence condition is deduced by going back to the original delayed control problem.

(b) The closed-loop representation of the open-loop optimal control is given through a coupled matrix-valued Riccati equation. The transformed stochastic optimal control problem with the new control operators can not be approximated by infinite dimensional control problems with bounded control operators, due to the lack of stochastic analytic tools. An integral operator-valued Riccati equation is constructed to overcome the difficulties caused by the new control operators, and inspired by this, the above coupled matrix-valued Riccati equation is obtained.

(c) The closed-loop solvability is assured by the solvability of a differential operator-valued Riccati equation. This is the first result for the closed-loop solvability of delayed stochastic optimal control problems. The difficulties are overcome through the introduction of the closed-loop strategy in decoupling forward delayed state equations and backward advanced adjoint equations, and sufficient conditions for the solvability of the Riccati equation are also provided. In addition, a clearer characterization of the closed-loop solvability is displayed by a coupled matrix-valued Riccati equation when going back to the original delayed control problem.

This paper is organized as follows. Section 2 formulates the optimal control problem for a stochastic differential delay system. Section 3 transforms it into an infinite dimensional control problem without delay. Section 4 derives necessary and sufficient conditions for the open-loop solvability. Section 5 presents the closed-loop representation of the open-loop optimal control. Section 6 ensures the closed-loop solvability under certain conditions. Finally Section 7 gives some concluding remarks.

2. Problem formulation. Suppose $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is generated by a one-dimensional standard Brownian motion $\{W(t)\}_{t\geqslant 0}$. $\mathbb{E}_t[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_t$, i.e. $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$. First we define the following spaces which will be used in this paper. Let F be a closed convex subset of $\mathbb{R}^n$, and E a real Banach space. Then, $L^\infty(F;E)$ denotes the Banach space consisting of E-valued functions $\phi(\cdot)$ such that $\sup_{t\in F}||\phi(t)||_E < \infty$, $H^1(F;E)$ denotes the Sobolev space consisting of square integrable functions with square integrable distributional derivatives $D_t\phi$, $L^2_{\mathbb{F}}(\Omega; C(F;E))$ denotes the Banach space consisting of E-valued $\mathbb{F}$-adapted continuous processes $\phi(\cdot)$ such that $\mathbb{E}[\sup_{t\in F}||\phi(t)||_E^2] < \infty$, $L^2_{\mathbb{F}}(F;E)$ denotes the Hilbert space consisting of $\mathbb{F}$-adapted processes $\phi(\cdot)$ such that $\mathbb{E}\int_F ||\phi(t)||_E^2 dt < \infty$. When $F = [a,b] \subseteq \mathbb{R}$, we simply denote $L^2(a,b;E)$ for $L^2([a,b];E)$ and other spaces are similar.

Let $||\cdot||_{H^1}$ and $\langle\cdot,\cdot\rangle_{H^1}$ denote the norm and the inner product in the Sobolev space $H^1(F;E)$, similar to other spaces. For simplicity, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product in the Euclidean space. E' denotes the dual space of E, and the symbol $\langle\cdot,\cdot\rangle_{E',E}$ is referred to as the duality pairing between E' and E. Given two real Hilbert space U_1 and U_2, $\mathscr{L}(U_1, U_2)$ denotes the real Banach space of all continuous linear maps, when $U_1 = U_2$, we write $\mathscr{L}(U_1)$ in place of $\mathscr{L}(U_1, U_2)$. Φ^*

denotes the adjoint operator of $\Phi \in \mathscr{L}(U_1, U_2)$. $\mathbb{S}^n$ is the space of all $n \times n$ symmetric matrices, I is the identity matrix with appropriate dimension or the identity map, and $\mathcal{R}$ is the operator range or the matrix range, if no ambiguity exists. The superscript † represents the Moore-Penrose inverse of vectors or matrices.

In this section, we formulate the stochastic optimal control problem.

For given finite time duration $T > 0$ and given constant time delay $\delta > 0$, let $A(d\theta)$ be $\mathbb{R}^{n\times n}$-valued finite measure on $[-\delta, 0]$ as follows:

$$\int_{[-\delta,0]} A(d\theta)\tilde{\varphi}(\theta) := \sum_{i=0}^{N} A_i \tilde{\varphi}(\theta_i) + \int_{-\delta}^{0} A^0(\theta)\tilde{\varphi}(\theta)d\theta, \tag{2.1}$$

with any square integrable function $\tilde{\varphi}(\cdot)$, and $-\delta = \theta_N < \theta_{N-1} < \cdots < \theta_1 < \theta_0 = 0$. A_i and A^0 represent the pointwise delay and the distributed delay, respectively. $B(d\theta)$ and $D(d\theta)$ are similar to (2.1), involving B_i, $B^0(\cdot)$ and D_i, $D^0(\cdot)$, respectively. The term about $C(d\theta)$ has the following form:

$$\int_{[-\delta,0]} C(d\theta)\tilde{\varphi}(\theta) := C_0\tilde{\varphi}(0) + \int_{-\delta}^{0} C^0(\theta)\tilde{\varphi}(\theta)d\theta. \tag{2.2}$$

For given $s \in [0, T)$, consider the following controlled linear SDDE:

$$\begin{cases} dX(t) = \displaystyle\int_{[-\delta,0]} \Big(A(d\theta)X_t(\theta) + B(d\theta)u_t(\theta)\Big)dt \\ \qquad\qquad + \displaystyle\int_{[-\delta,0]} \Big(C(d\theta)X_t(\theta) + D(d\theta)u_t(\theta)\Big)dW(t), \ \ t \in [s, T], \\ X(s) = x, \quad X(t) = \varphi(t-s), \quad t \in [s-\delta, s), \\ u(t) = \psi(t-s), \quad t \in [s-\delta, s], \end{cases} \tag{2.3}$$

along with the cost functional as follows:

$$\begin{aligned} J(s, x, \varphi(\cdot), \psi(\cdot); u(\cdot)) = \mathbb{E}\Big\{ \int_s^T \Big[&\int_{[-\delta,0]^2} \langle Q(t, d\theta d\theta')X_t(\theta), X_t(\theta')\rangle \\ &+ 2\langle S(t, d\theta d\theta')X_t(\theta), u_t(\theta')\rangle + \langle R(t, d\theta d\theta')u_t(\theta), u_t(\theta')\rangle \Big] dt \\ &+ \int_{[-\delta,0]^2} \langle G(d\theta d\theta')X_T(\theta), X_T(\theta')\rangle \Big\}. \end{aligned} \tag{2.4}$$

Here, $X(\cdot)$ is the state and $u(\cdot) \in L^2_{\mathbb{F}}(s, T; \mathbb{R}^m)$ is the control. x is the initial state, $\varphi(\cdot) \in L^2(-\delta, 0; \mathbb{R}^n)$ and $\psi(\cdot) \in L^2(-\delta, 0; \mathbb{R}^m)$ are the initial trajectories of the state and the control, respectively. $X_t(\cdot) := X(t + \cdot)$ and $u_t(\cdot) := u(t + \cdot)$, represent the past trajectories of the state and the control. In the cost functional (2.4), $Q(t, d\theta d\theta')$ and $S(t, d\theta d\theta')$ are also finite measures, involving $Q_{00}(\cdot), Q_{10}(\cdot, \cdot)$, $Q_{11}(\cdot, \cdot, \cdot)$ and $S_{00}(\cdot), S_{01}(\cdot, \cdot), S_{10}(\cdot, \cdot), S_{11}(\cdot, \cdot, \cdot)$, respectively:

$$\begin{aligned} &\int_{[-\delta,0]^2} \langle Q(t, d\theta d\theta')\tilde{\varphi}(\theta), \tilde{\varphi}(\theta')\rangle := \int_{[-\delta,0]^2} \big\langle Q_{11}(t, \theta, \theta')\tilde{\varphi}(\theta), \tilde{\varphi}(\theta')\big\rangle d\theta' d\theta \\ &\quad + \big\langle Q_{00}(t)\tilde{\varphi}(0), \tilde{\varphi}(0)\big\rangle + 2\int_{-\delta}^{0} \big\langle Q_{10}(t, \theta)^\top \tilde{\varphi}(\theta), \tilde{\varphi}(0)\big\rangle d\theta, \ \forall \tilde{\varphi} \in L^2(-\delta, 0; \mathbb{R}^n), \\ &\int_{[-\delta,0]^2} \langle S(t, d\theta d\theta')\tilde{\varphi}(\theta), \tilde{\psi}(\theta')\rangle := \big\langle S_{00}(t)\tilde{\varphi}(0), \tilde{\psi}(0)\big\rangle \\ &+ \int_{-\delta}^{0} \big\langle S_{01}(t, \theta)\tilde{\varphi}(\theta), \tilde{\psi}(0)\big\rangle d\theta + \int_{-\delta}^{0} \big\langle S_{10}(t, \theta)^\top \tilde{\psi}(\theta), \tilde{\varphi}(0)\big\rangle d\theta \\ &+ \int_{[-\delta,0]^2} \big\langle S_{11}(t, \theta, \theta')\tilde{\varphi}(\theta), \tilde{\psi}(\theta')\big\rangle d\theta' d\theta, \ \forall \tilde{\varphi} \in L^2(-\delta, 0; \mathbb{R}^n), \tilde{\psi} \in L^2(-\delta, 0; \mathbb{R}^m), \end{aligned}$$

$R(t, d\theta d\theta')$ and $G(d\theta d\theta')$ are similar to $Q(t, d\theta d\theta')$, involving $R_{00}(\cdot), R_{10}(\cdot, \cdot), R_{11}(\cdot, \cdot, \cdot)$ and $G_{00}, G_{10}(\cdot), G_{11}(\cdot, \cdot)$. In the above, $A_i, C_0, G_{00} \in \mathbb{R}^{n\times n}$, $B_i, D_i \in \mathbb{R}^{n\times m}$, $i = 0, \cdots, N$, $A^0(\cdot), B^0(\cdot), C^0(\cdot), D^0(\cdot), Q_{00}(\cdot), Q_{10}(\cdot, \cdot), Q_{11}(\cdot, \cdot, \cdot), S_{00}(\cdot), S_{01}(\cdot, \cdot), S_{10}(\cdot, \cdot), S_{11}(\cdot, \cdot, \cdot), R_{00}(\cdot)$, $R_{10}(\cdot, \cdot), R_{11}(\cdot, \cdot, \cdot), G_{10}(\cdot), G_{11}(\cdot, \cdot)$ are matrix-valued functions of appropriate dimensions.

Let us assume the following:

(A1) The coefficients of the state equation (2.3) satisfy the following assumptions:

$$A^0(\cdot), C^0(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B^0(\cdot), D^0(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}).$$

(A2) The coefficients of the cost functional (2.4) satisfy the following assumptions:

$$\begin{aligned}
&Q_{00}(\cdot) \in L^\infty(0,T;\mathbb{S}^n), \quad Q_{10}(\cdot,\cdot) \in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{n\times n}),\\
&Q_{11}(\cdot,\cdot,\cdot) \in L^\infty([0,T]\times[-\delta,0]\times[-\delta,0];\mathbb{R}^{n\times n}), \quad S_{00}(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times n}),\\
&S_{01}(\cdot,\cdot) \in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{m\times n}), \quad S_{10}(\cdot,\cdot) \in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{m\times n}),\\
&S_{11}(\cdot,\cdot,\cdot) \in L^\infty([0,T]\times[-\delta,0]\times[-\delta,0];\mathbb{R}^{m\times n}), \quad R_{00}(\cdot) \in L^\infty(0,T;\mathbb{S}^m),\\
&R_{10}(\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{m\times m}), R_{11}(\cdot,\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0]\times[-\delta,0];\mathbb{R}^{m\times m}),\\
&G_{10}(\cdot,\cdot) \in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{n\times n}), \quad G_{11}(\cdot) \in L^2(-\delta,0;\mathbb{R}^{n\times n}), \quad G_{00} \in \mathbb{S}^n.\\
&Q_{11}(t,\theta,\theta')^\top = Q_{11}(t,\theta',\theta),\ R_{11}(t,\theta,\theta')^\top = R_{11}(t,\theta',\theta),\ G_{11}(\theta,\theta')^\top = G_{11}(\theta',\theta).
\end{aligned}$$

We choose the product space $\mathfrak{M} := \mathbb{R}^n \times L^2(-\delta,0;\mathbb{R}^n)$ as the space of initial data, which is a Hilbert space endowed with inner product and norm

$$\langle x,y\rangle_{\mathfrak{M}} := \langle x^0,y^0\rangle + \int_{-\delta}^0 \langle x^1(\theta),y^1(\theta)\rangle d\theta, \quad \text{and} \quad ||x||_{\mathfrak{M}} := \langle x,x\rangle_{\mathfrak{M}}^{\frac{1}{2}},$$
$$\forall x = \begin{pmatrix} x^0\\ x^1\end{pmatrix}, \quad y = \begin{pmatrix} y^0\\ y^1\end{pmatrix}, \quad x^0,y^0 \in \mathbb{R}^n, x^1,y^1 \in L^2(-\delta,0;\mathbb{R}^n).$$

Under Assumptions (A1)–(A2), for any initial data $(s,x,\varphi,\psi) \in [0,T)\times\mathfrak{M}\times L^2(-\delta,0;\mathbb{R}^m)$ and any admissible control $u(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, by the Picard iteration method or by Theorem 2.1 ([25], Chapter II), the SDDE (2.3) admits a unique solution $X(\cdot)\equiv X(\cdot;s,x,\varphi,\psi,u(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathbb{R}^n))$, therefore the cost functional (2.4) is meaningful.

Problem (P). For any $(s,x,\varphi,\psi) \in [0,T)\times\mathfrak{M}\times L^2(-\delta,0;\mathbb{R}^m)$, to find a $\bar{u}(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (2.3) is satisfied and

$$J(s,x,\varphi(\cdot),\psi(\cdot);\bar{u}(\cdot)) = \inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)} J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot)) := V(s,x,\varphi(\cdot),\psi(\cdot)).$$

Any $\bar{u}(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ that achieves the above infimum is called an *optimal control* for the initial data (s,x,φ,ψ), and the corresponding solution $\bar{X}(\cdot) \equiv X(\cdot\,;s,x,\varphi,\psi,\bar{u}(\cdot))$ is called the *optimal state*. The function $V(\cdot,\cdot,\cdot,\cdot)$ is called the *value function* of Problem (P).

3. Problem transformation.

In this section, inspired by [6] and [12], we study Problem (P) by a control problem without delay, containing a new control operator.

Define the C_0-semigroup $\Phi(\cdot)$ as follows:

$$\begin{aligned}
&\Phi(t) : \mathfrak{M} \longrightarrow \mathfrak{M}\\
&\xi \mapsto \begin{pmatrix} x(t)\\ x_t(\cdot)\end{pmatrix}, \quad \forall \xi := \begin{pmatrix} x\\ \varphi\end{pmatrix} \in \mathfrak{M},
\end{aligned} \tag{3.1}$$

where $x(\cdot) \equiv x(\cdot\,;s,x,\varphi)$ is the solution to the following equation:

$$\begin{cases} \dot{x}(t) = \int_{[-\delta,0]} A(d\theta)x_t(\theta),\ \text{a.e. } t \in [0,T],\\ x(0) = x,\ x(t) = \varphi(t),\ t \in [-\delta,0),\end{cases}$$

with $x_t(\cdot) := x(t+\cdot)$. The generator of $\Phi(\cdot)$ is defined as

$$\begin{aligned}
&\tilde{A} : \mathscr{D}(\tilde{A}) \longrightarrow \mathfrak{M}\\
&\xi \mapsto \begin{pmatrix} \int_{[-\delta,0]} A(d\theta)\varphi(\theta)\\ \dot{\varphi}(\cdot)\end{pmatrix}, \quad \forall \xi \in \mathscr{D}(\tilde{A}),
\end{aligned} \tag{3.2}$$

and its domain is $\mathscr{D}(\tilde{A}) = \{\xi = (x^\top,\varphi^\top)^\top \in \mathfrak{M} |\ \varphi(\cdot) \in H^1(-\delta,0;\mathbb{R}^n), x = \varphi(0)\}$. As mentioned in [6], $\mathscr{D}(\tilde{A})$ is dense in $\mathfrak{M}$ and is a Banach space endowed with the norm $||\xi||_{\mathscr{D}(\tilde{A})} := ||\varphi(\cdot)||_{H^1}$. Denote $\mathfrak{L} := L^2(-\delta,0;\mathbb{R}^m)$ and define the following operators:

$$\tilde{B}:\mathfrak{L}\longrightarrow\mathfrak{M}\qquad\qquad \tilde{D}:\mathfrak{L}\longrightarrow\mathfrak{M}\qquad\qquad \tilde{C}:\mathfrak{M}\longrightarrow\mathfrak{M}$$

$$\psi\mapsto\begin{pmatrix}\int_{[-\delta,0]}B(d\theta)\psi(\theta)\\ 0\end{pmatrix},\ \psi\mapsto\begin{pmatrix}\int_{[-\delta,0]}D(d\theta)\psi(\theta)\\ 0\end{pmatrix},\ \begin{pmatrix}x\\ \varphi\end{pmatrix}\mapsto\begin{pmatrix}C_0x+\int_{-\delta}^0C^0(\theta)\varphi(\theta)d\theta\\ 0\end{pmatrix}. \tag{3.3}$$

Then, $\tilde{C}\in\mathscr{L}(\mathfrak{M})$, but $\tilde{B},\tilde{D}\notin\mathscr{L}(\mathfrak{L},\mathfrak{M})$. Thus, we can write (2.3) in $\mathbb{R}^n$ as the following *stochastic evolution equation* (SEE) in $\mathfrak{M}$:

$$\begin{cases} d\mathbf{X}(t)=[\tilde{A}\mathbf{X}(t)+\tilde{B}u_t]dt+[\tilde{C}\mathbf{X}(t)+\tilde{D}(t)u_t]dW(t), & t\in[s,T],\\ \mathbf{X}(s)=\xi=\begin{pmatrix}x\\ \varphi\end{pmatrix},\ u(t)=\psi(t-s), \quad t\in[s-\delta,s]. \end{cases} \tag{3.4}$$

By Theorem 3.14 in [22], the SEE (3.4) has a unique solution. If we regard $\mathbf{X}(\cdot)$ as the new state, then (3.4) does not contain state delay. Before dealing with control delay, we give the following result to illustrate the equivalence of (2.3) and (3.4).

LEMMA 3.1. *Let* (A1)–(A2) *hold. For all* $\xi\in\mathfrak{M}$, $\psi(\cdot)\in\mathfrak{L}$, $u(\cdot)\in L^2(s,T;\mathbb{R}^m)$, *assume that* $X(\cdot)$ *is the solution to* (2.3)*. Then,* $\mathbf{X}(\cdot)$ *defined as* $\mathbf{X}(t):=\begin{pmatrix}X(t)\\ X_t(\cdot)\end{pmatrix}$, *is the mild solution to* (3.4)*, i.e.*

$$\mathbf{X}(t)=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde{B}u_rdr+\int_s^t\Phi(t-r)\big[\tilde{C}\mathbf{X}(r)+\tilde{D}u_r\big]dW(r),\, t\in[s,T]. \tag{3.5}$$

Furthermore, there exists a constant $M>0$ *such that*

$$\mathbb{E}\Big[\sup_{s\leqslant t\leqslant T}||\mathbf{X}(t)||^2_{\mathfrak{M}}\Big]\le M\Big[|x|^2+\int_{-\delta}^0(|\varphi(\theta)|^2+|\psi(\theta)|^2)d\theta+\mathbb{E}\int_s^T|u(r)|^2dr\Big].$$

The proof is similar to Theorem 2.3 in [8], and thus is omitted here.

Remark 3.2. From Lemma 3.1, the SDDE (2.3) is equivalent to the SEE (3.5). When $C_0,C^0(\theta),D_i,D^0(\theta)$ depend on t, Lemma 3.1 holds, $i=0,\cdots,N$. When (2.2) contains multiple pointwise delays, Lemma 3.1 also holds, see Pages 941–943 in [8].

Next we deal with control delay, introduce the semigroup of left translation:

$$\mathcal{L}(t):\mathfrak{L}\longrightarrow\mathfrak{L}$$

$$\big[\mathcal{L}(t)Y\big](\theta):=\begin{cases}\begin{cases}Y(t+\theta), & -\delta\le\theta\le -t,\\ 0, & -t<\theta\le 0,\end{cases} & \text{if } t\le\delta,\\ 0, \qquad -\delta\le\theta\le 0, & \text{if } t>\delta.\end{cases} \tag{3.6}$$

Its generator is given by $\mathcal{A}:\mathscr{D}(\mathcal{A})\longrightarrow\mathfrak{L}$, $\mathcal{A}Y:=\frac{dY}{d\theta}$, $\forall Y\in\mathscr{D}(\mathcal{A})$. The domain $\mathscr{D}(\mathcal{A})=\big\{Y\in H^1(-\delta,0;\mathbb{R}^m)|\ Y$ is absolutely continuous and $Y(0)=0\big\}$, is a Banach space endowed with the norm $||\cdot||_{H^1}$. Denote $V:=H^1(-\delta,0;\mathbb{R}^m)$, let V' be the dual of V, and consider the following evolution equation:

$$\mathbf{Y}_t=\mathcal{L}(t-s)\psi+\int_s^t\mathcal{L}(t-r)\Delta u(r)dr, \quad t\in[s,T], \tag{3.7}$$

with the bounded linear operator $\Delta:\mathbb{R}^m\longrightarrow V'$, $\langle\Delta u,w\rangle_{V',V}:=\langle u,w(0)\rangle$, $\forall\ u\in\mathbb{R}^m,\ w\in V$. Then, by Lemma 1.1 in [12], (3.7) is well-defined and

$$\mathbf{Y}_t(\theta)=\begin{cases}\begin{cases}u(t+\theta), & s-t<\theta\le 0,\\ \psi(\theta+t-s), & -\delta\le\theta\le s-t,\end{cases} & \text{if } t-s\le\delta,\\ u(t+\theta), \qquad -\delta\le\theta\le 0, & \text{if } t-s>\delta.\end{cases} \tag{3.8}$$

By (3.8), we get $\mathbf{Y}_t(\theta)=u_t(\theta)$ for almost everywhere $\theta\in[-\delta,0]$ and all $t\in[s,T]$.

Therefore, (3.5) can be written as the following formula, equivalent to (2.3):

$$\begin{cases}\mathbf{X}(t)=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde{B}\mathbf{Y}_rdr+\int_s^t\Phi(t-r)[\tilde{C}\mathbf{X}(r)+\tilde{D}\mathbf{Y}_r]dW(r),\, t\in[s,T],\\ \mathbf{Y}_t=\mathcal{L}(t-s)\psi+\int_s^t\mathcal{L}(t-r)\Delta u(r)dr, \quad t\in[s,T].\end{cases} \tag{3.9}$$

Denote $\mathfrak{Z} := \mathfrak{M}\times\mathfrak{L}$, for any $z=\begin{pmatrix}\xi\\ \psi\end{pmatrix}$, $z_1=\begin{pmatrix}\xi_1\\ \psi_1\end{pmatrix}$ and $z_2=\begin{pmatrix}\xi_2\\ \psi_2\end{pmatrix}\in\mathfrak{Z}$, $\|z\|_{\mathfrak{Z}} := \big[\|\xi\|^2_{\mathfrak{M}}+\|\psi\|^2_{\mathfrak{L}}\big]^{\frac{1}{2}}$, $\langle z_1,z_2\rangle_{\mathfrak{Z}}:=\langle \xi_1,\xi_2\rangle_{\mathfrak{M}}+\langle\psi_1,\psi_2\rangle_{\mathfrak{L}}$. Define the following C_0-semigroup:

$$\mathbf{T}(t):\mathfrak{Z}\longrightarrow\mathfrak{Z}$$
$$\mathbf{T}(t)\begin{pmatrix}\xi\\ \psi\end{pmatrix} := \begin{bmatrix}\Phi(t)\xi+\int_0^t\Phi(t-r)\tilde{B}\mathcal{L}(r)\psi dr\\ \mathcal{L}(t)\psi\end{bmatrix},$$

and $\mathbf{Z}_0:=\begin{pmatrix}\xi\\ \psi\end{pmatrix}$, $\mathbf{Z}(\cdot):=\begin{pmatrix}\mathbf{X}(\cdot)\\ \mathbf{Y}_\cdot\end{pmatrix}$, $\mathbf{B}:=\begin{pmatrix}0\\ \Delta\end{pmatrix}$, $\mathbf{C}:=\begin{pmatrix}\tilde{C} & \tilde{D}\\ 0 & 0\end{pmatrix}$. Then, (3.9) can be written as

$$\mathbf{Z}(t)=\mathbf{T}(t-s)\mathbf{Z}_0+\int_s^t\mathbf{T}(t-r)\mathbf{B}u(r)dr+\int_s^t\mathbf{T}(t-r)\mathbf{C}\mathbf{Z}(r)dW(r). \tag{3.10}$$

Noting $\mathbf{C}\notin\mathscr{L}(\mathfrak{Z})$, $\mathbf{B}$ maps $\mathbb{R}^m$ to $\mathfrak{M}\times V'$ out $\mathfrak{Z}$, thus $\mathbf{B}\notin\mathscr{L}(\mathbb{R}^m,\mathfrak{Z})$, the above integration is not defined in $\mathfrak{Z}$, (3.10) is just a formal expression and it actually means (3.9).

Now we have transformed the original delayed state equation (2.3) into the new state equation (3.10) (or (3.9)), containing neither state delay nor control delay.

Next we rewrite the cost functional (2.4) by $\mathbf{Z}(\cdot)$ and $u(\cdot)$, before that we define some bounded linear operators. Recalling $\mathfrak{L}:=L^2(-\delta,0;\mathbb{R}^m)$, we also denote $L^2(-\delta,0;\mathbb{R}^n)$ by $\mathfrak{L}$ for ease of writing, and the dimension depends on the specific situation. Denote $\tilde{\kappa}_{00}(t)\tilde{x}=\kappa_{00}(t)\tilde{x}$, $\tilde{\kappa}_{01}(t)\tilde{\varphi}:=\int_{-\delta}^0\kappa_{01}(t,\theta)\tilde{\varphi}(\theta)d\theta$, $(\tilde{\kappa}_{10}(t)\tilde{x})(\cdot):=\kappa_{10}(t,\cdot)\tilde{x}$, $(\tilde{\kappa}_{11}(t)\tilde{\varphi})(\cdot):=\int_{-\delta}^0\kappa_{11}(t,\theta,\cdot)\tilde{\varphi}(\theta)d\theta$, for any $\tilde{x}\in\mathbb{R}^d,\tilde{\varphi}\in\mathfrak{L}, d=n,m$, where $\kappa=Q,S,R,G$, $Q_{01}(t,\theta)=Q_{10}(t,\theta)^\top$, $R_{01}(t,\theta)=R_{10}(t,\theta)^\top$, $G_{01}(\theta)=G_{10}(\theta)^\top$. Then, $\tilde{Q}_{01}(t)^*=\tilde{Q}_{10}(t)$, $\tilde{R}_{01}(t)^*=\tilde{R}_{10}(t)$, $\tilde{G}_{01}^*=\tilde{G}_{10}$. Notice that $\tilde{S}_{01}(t)^*=\tilde{S}_{10}(t)$ is not always true. Let

$$\tilde{Q}(t):=\begin{bmatrix}\tilde{Q}_{00}(t) & \tilde{Q}_{01}(t)\\ \tilde{Q}_{10}(t) & \tilde{Q}_{11}(t)\end{bmatrix},\ \tilde{S}(t):=\begin{bmatrix}\tilde{S}_{00}(t) & \tilde{S}_{01}(t)\\ \tilde{S}_{10}(t) & \tilde{S}_{11}(t)\end{bmatrix},\ \tilde{R}(t):=\begin{bmatrix}\tilde{R}_{00}(t) & \tilde{R}_{01}(t)\\ \tilde{R}_{10}(t) & \tilde{R}_{11}(t)\end{bmatrix},\ \tilde{G}:=\begin{bmatrix}\tilde{G}_{00} & \tilde{G}_{01}\\ \tilde{G}_{10} & \tilde{G}_{11}\end{bmatrix}.$$

Then, we rewrite the cost functional (2.4) as follows

$$\begin{aligned} J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot))=&\,\mathbb{E}\int_s^T\Big[\big\langle\tilde{Q}(t)\mathbf{X}(t),\mathbf{X}(t)\big\rangle+2\Big\langle\tilde{S}(t)\mathbf{X}(t),\begin{pmatrix}u(t)\\ \mathbf{Y}_t\end{pmatrix}\Big\rangle\\ &+\Big\langle\tilde{R}(t)\begin{pmatrix}u(t)\\ \mathbf{Y}_t\end{pmatrix},\begin{pmatrix}u(t)\\ \mathbf{Y}_t\end{pmatrix}\Big\rangle\Big]dt+\mathbb{E}\big\langle\tilde{G}\mathbf{X}(T),\mathbf{X}(T)\big\rangle. \end{aligned}\tag{3.11}$$

In the above, $\langle\cdot,\cdot\rangle$ has the different meaning.

Define

$$\tilde{S}_0(t):=\big[\tilde{S}_{00}(t)\quad\tilde{S}_{01}(t)\big],\ \tilde{S}_1(t):=\big[\tilde{S}_{10}(t)\quad\tilde{S}_{11}(t)\big],\ \mathbf{S}(t):=\big[\tilde{S}_0(t)\quad\tilde{R}_{01}(t)\big],$$
$$\mathbf{Q}(t):=\begin{bmatrix}\tilde{Q}(t) & \tilde{S}_1(t)^*\\ \tilde{S}_1(t) & \tilde{R}_{11}(t)\end{bmatrix},\quad \mathbf{G}:=\begin{bmatrix}\tilde{G} & 0\\ 0 & 0\end{bmatrix},\quad \mathbf{R}(t):=\tilde{R}_{00}(t).$$

Then, we rewrite (3.11) like this:

$$\begin{aligned} J(s,\mathbf{Z}_0;u(\cdot))=&\,\mathbb{E}\int_s^T\Big[\big\langle\mathbf{Q}(t)\mathbf{Z}(t),\mathbf{Z}(t)\big\rangle_{\mathfrak{Z}}+2\big\langle\mathbf{S}(t)\mathbf{Z}(t),u(t)\big\rangle\\ &+\big\langle\mathbf{R}(t)u(t),u(t)\big\rangle\Big]dt+\mathbb{E}\big\langle\mathbf{G}\mathbf{Z}(T),\mathbf{Z}(T)\big\rangle_{\mathfrak{Z}}, \end{aligned}\tag{3.12}$$

thus we transform Problem (P) into a linear quadratic problem associated with (3.10) (or (3.9)) and (3.12), and we formulate it specifically as follows.

Problem (EP). For any $(s,\mathbf{Z}_0)\in[0,T)\times\mathfrak{Z}$, to find a $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (3.10) (or (3.9)) is satisfied and

$$J(s,\mathbf{Z}_0;\bar{u}(\cdot))=\inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)}J(s,\mathbf{Z}_0;u(\cdot)):=V(s,\mathbf{Z}_0). \tag{3.13}$$

Similarly, any $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ that achieves the above infimum is called an *optimal control* for the initial pair $(s,\mathbf{Z}_0)$, and the corresponding solution $\bar{\mathbf{Z}}(\cdot)$ is called the *optimal state*. The function $V(\cdot,\cdot)$ is called the *value function* of Problem (EP).

Remark 3.3. By (3.7), (3.8), (3.9) and Remark 3.2, Problem (P) is equivalent to Problem (EP). When $C_0, C^0(\theta), D_i, D^0(\theta)$ depend on t, the equivalence also holds, $i=0,\cdots,N$. We transform the delayed finite dimensional Problem (P) into the infinite dimensional Problem (EP) without delay, containing the new control operator $\mathbf{B}$. It is worth mentioning that the unboundedness of $\mathbf{B}$ is as high as that studied by [15,16], but its domain does not have a relation to that of the semigroup generator. Therefore, the existing approaches in the literature do not apply. In the rest section, we will take some new methods to address the unboundedness of the control operator.

4. Open-loop solvability. In this section, we define the open-loop solvability for Problem (P) by the transformed Problem (EP), and assure it by the solvability of a constrained forward-backward stochastic evolution system and a convexity condition. Finally we turn back to the original Problem (P) and explore its open-loop solvability.

First we give the definition of the open-loop solvability for Problem (P).

DEFINITION 4.1. Problem (P) *is said to be*

(i) (uniquely) open-loop solvable at initial data $(s,x,\varphi,\psi)\in[0,T]\times\mathfrak{Z}$, *if there exists a (unique)* $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ *satisfying (3.13).*

(ii) (uniquely) open-loop solvable at some $s\in[0,T)$, *if for any* $(x,\varphi,\psi)\in\mathfrak{Z}$, *there exists a (unique)* $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ *satisfying (3.13).*

(iii) (uniquely) open-loop solvable on $[s,T)$, *if it is (uniquely) open-loop solvable at all* $t\in[s,T)$.

Next we give the necessary and sufficient condition of the open-loop solvability.

THEOREM 4.2. *Let* (A1)–(A2) *hold. For any given initial data* $(s,x,\varphi,\psi)\in[0,T)\times\mathfrak{Z}$, $\bar{u}(\cdot)$ *is an open-loop optimal control of* Problem (P) *if and only if the following two conditions hold:*

(i) (Stationarity condition)

$$\tilde{S}_0(t)\bar{\mathbf{X}}(t)+\tilde{R}_{01}(t)\bar{\mathbf{Y}}_t+\tilde{R}_{00}(t)\bar{u}(t)+[p_2(t)](0)=0,\quad \text{a.e. a.s.,} \tag{4.1}$$

where $(\bar{\mathbf{X}}(\cdot),\bar{\mathbf{Y}}_{\cdot},p_1(\cdot),k_1(\cdot),p_2(\cdot),k_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{M}))\times L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{M}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{M})\times L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$ *is the solution to the following forward-backward SEE:*

$$\begin{cases} (a)\ \bar{\mathbf{X}}(t)=\Phi(t-s)\xi+\displaystyle\int_s^t\Phi(t-r)\tilde{B}\bar{\mathbf{Y}}_r dr \\ \qquad\qquad +\displaystyle\int_s^t\Phi(t-r)\Big(\tilde{C}\bar{\mathbf{X}}(r)+\tilde{D}\bar{\mathbf{Y}}_r\Big)dW(r),\quad t\in[s,T], \\ (b)\ \bar{\mathbf{Y}}_t=\mathcal{L}(t-s)\psi+\displaystyle\int_s^t\mathcal{L}(t-r)\Delta\bar{u}(r)dr,\quad t\in[s,T], \\ (c)\ p_1(t)=\Phi(T-t)^*\tilde{G}\bar{\mathbf{X}}(T)+\displaystyle\int_t^T\Phi(r-t)^*\big[\tilde{C}^*k_1(r)+\tilde{Q}(r)\bar{\mathbf{X}}(r)+\tilde{S}_0(r)^*\bar{u}(r) \\ \qquad\qquad +\tilde{S}_1(r)^*\bar{\mathbf{Y}}_r\big]dr-\displaystyle\int_t^T\Phi(r-t)^*k_1(r)dW(r),\quad t\in[s,T], \\ (d)\ [p_2(t)](\theta)=\displaystyle\int_t^{T\wedge(t+\delta+\theta)}\big[\tilde{S}_1(r)\bar{\mathbf{X}}(r)+\tilde{R}_{01}(r)^*\bar{u}(r)+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r\big](t+\theta-r)dr \\ \qquad\qquad +\displaystyle\int_{[-\delta,0]}\Big(B(d\beta)^\top[p_1(t+\theta-\beta)]^0+D(d\beta)^\top[k_1(t+\theta-\beta)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\beta) \\ \qquad\qquad -\displaystyle\int_t^{T\wedge(t+\delta+\theta)}[k_2(r)](t+\theta-r)dW(r),\quad t\in[s,T],\ \theta\in[-\delta,0], \end{cases} \tag{4.2}$$

with $\xi=\big(x^\top,\varphi^\top\big)^\top$. $[p_1(r)]^0,[k_1(r)]^0\in\mathbb{R}^n$ *denote the* $\mathbb{R}^n$ *components of* $p_1(r)$ *and* $k_1(r)$.

(ii) (Convexity condition)

$$J(s,0;u^0(\cdot)) \geqslant 0,\ \forall u^0(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m), \tag{4.3}$$

where $(\mathbf{X}^0(\cdot),\mathbf{Y}^0_\cdot)$ *is the solution to the following integral equation:*

$$\begin{cases} \mathbf{X}^0(t) = \int_s^t \Phi(t-r)\tilde{B}\mathbf{Y}^0_r dr + \int_s^t \Phi(t-r)\Big(\tilde{C}\mathbf{X}^0(r) + \tilde{D}\mathbf{Y}^0_r\Big)dW(r), \quad t\in[s,T], \\ \mathbf{Y}^0_t = \int_s^t \mathcal{L}(t-r)\Delta u^0(r)dr, \qquad t\in[s,T]. \end{cases}$$

Proof. We split the proof into three steps as follows.

Step 1: For given $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, show that the forward-backward SEE (4.2) admits a unique solution.

By Theorem 4.10 in [22], $(p_1(\cdot),k_1(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{M}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{M})$. It remains to prove that (4.2)(d) admits a unique solution $(p_2(\cdot),k_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$ for given $(\bar{\mathbf{X}}(\cdot),\bar{\mathbf{Y}}_\cdot,\, p_1(\cdot),k_1(\cdot))$. Notice that $\tilde{B},\tilde{D}\in\mathscr{L}(\mathscr{D}(\mathcal{A}),\mathfrak{M})$. Then, for any $\kappa\in\mathscr{D}(\mathcal{A})$,

$$\begin{aligned} &\Big\langle \int_t^T \mathcal{L}(r-t)^*\tilde{B}^*p_1(r)dr,\kappa\Big\rangle_{\langle\mathscr{D}(\mathcal{A})',\mathscr{D}(\mathcal{A})\rangle} = \int_t^T \Big\langle \tilde{B}^*p_1(r),\mathcal{L}(r-t)\kappa\Big\rangle_{\langle\mathscr{D}(\mathcal{A})',\mathscr{D}(\mathcal{A})\rangle}dr \\ &= \int_t^T \langle p_1(r),\tilde{B}\mathcal{L}(r-t)\kappa\rangle_{\mathfrak{M}}dr = \int_{-\delta}^0 \Big\langle \int_{[-\delta,0]} B(d\theta)^\top[p_1(r+t-\theta)]^0\mathbf{1}_{[t+r-T,r]}(\theta),\kappa(r)\Big\rangle dr, \end{aligned}$$

it follows that

$$\Big(\int_t^T \mathcal{L}(r-t)^*\tilde{B}^*p_1(r)dr\Big)(\theta) = \int_{[-\delta,0]} B(d\beta)^\top[p_1(t+\theta-\beta)]^0\mathbf{1}_{[t+\theta-T,\theta]}(\beta). \tag{4.4}$$

Similarly, we have

$$\Big(\int_t^T \mathcal{L}(r-t)^*\tilde{D}^*k_1(r)dr\Big)(\theta) = \int_{[-\delta,0]} D(d\beta)^\top[k_1(t+\theta-\beta)]^0\mathbf{1}_{[t+\theta-T,\theta]}(\beta). \tag{4.5}$$

By (3.6) and Lemma 3.3 in [7], (4.2)(d) is equivalent to the following backward SEE:

$$\begin{aligned} \tilde{p}_2(t) = &\int_t^T \mathcal{L}(r-t)^*\Big[\tilde{B}^*p_1(r)+\tilde{D}^*k_1(r)+\tilde{S}_1(r)\bar{\mathbf{X}}(r)+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r \\ &+\tilde{R}_{01}(r)^*\bar{u}(r)\Big]dr - \int_t^T \mathcal{L}(r-t)^*\tilde{k}_2(r)dW(r),\ t\in[s,T]. \end{aligned} \tag{4.6}$$

Next we would like to prove that (4.6) admits a unique solution $(\tilde{p}_2(\cdot),\tilde{k}_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$, and we only need to prove the existence. Denote

$$\tilde{p}_2(t) := \mathbb{E}_t\Big[\int_t^T \mathcal{L}(r-t)^*\Big(\tilde{B}^*p_1(r)+\tilde{D}^*k_1(r)+\tilde{S}_1(r)\bar{\mathbf{X}}(r)+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r+\tilde{R}_{01}(r)^*\bar{u}(r)\Big)dr\Big].$$

Then, we have

$$\begin{aligned} \tilde{p}_2(t) = &\mathbb{E}_t\Big[\int_{[-\delta,0]}\Big(D(d\beta)^\top[k_1(t+\cdot-\beta)]^0+B(d\beta)^\top[p_1(t+\cdot-\beta)]^0\Big)\mathbf{1}_{[t+\cdot-T,\cdot]}(\beta)\Big] \\ &+\mathbb{E}_t\Big[\int_t^T \mathcal{L}(r-t)^*\Big(\tilde{S}_1(r)\bar{\mathbf{X}}(r)+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r+\tilde{R}_{01}(r)^*\bar{u}(r)\Big)dr\Big] := \mathbf{I}(t)+\mathbf{II}(t). \end{aligned}$$

Let $L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathfrak{L}))$ be the Banach space of all strongly $\mathscr{B}([s,T])\otimes\mathscr{B}([s,T])\otimes\mathcal{F}_T$-measurable functions $h:[s,T]^2\times\Omega\to\mathfrak{L}$, satisfying that for $r\in[s,T]$, $h(r,\cdot)$ is $\mathbb{F}$-adapted and $\mathbb{E}\int_s^T\int_s^T||h(r,\beta)||^2_{L^2}d\beta dr<\infty$. Notice that $\tilde{S}_1(\cdot)\bar{\mathbf{X}}(\cdot)+\tilde{R}_{11}(\cdot)\bar{\mathbf{Y}}_\cdot+\tilde{R}_{01}(\cdot)^*\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathfrak{L})$. Then, by Corollary 2.149 in [22], there exists $h(\cdot,\cdot)\in L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathfrak{L}))$ such that

$$\mathbf{II}(t) = \int_t^T \mathcal{L}(r-t)^*\Big\{\Big(\tilde{S}_1(r)\bar{\mathbf{X}}(r)+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r+\tilde{R}_{01}(r)^*\bar{u}(r)\Big)-\int_t^r h(r,\beta)dW(\beta)\Big\}dr,\ t\in[s,T],$$

which yields

$$\mathbf{II}(t) = \int_t^T \mathcal{L}(r-t)^*\Big(\tilde{S}_1(r)\bar{\mathbf{X}}(r) + \tilde{R}_{11}(r)\bar{\mathbf{Y}}_r + \tilde{R}_{01}(r)^*\bar{u}(r)\Big)dr$$
$$- \int_t^T \mathcal{L}(r-t)^* \int_r^T \mathcal{L}(\beta - r)^* h(\beta, r) d\beta dW(r).$$

For $\mathbf{I}(t)$, we have

$$[\mathbf{I}(t)](\theta) = \mathbb{E}_t\Bigg[\sum_{i=0}^{N}\Big(D_i^\top[k_1(t+\theta-\theta_i)]^0 + B_i^\top[p_1(t+\theta-\theta_i)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\theta_i)$$
$$+\int_{-\delta}^0 \Big(D^0(\beta)^\top[k_1(t+\theta-\beta)]^0 + B^0(\beta)^\top[p_1(t+\theta-\beta)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\beta)d\beta\Bigg]. \tag{4.7}$$

Since $[p_1(\cdot)]^0, [k_1(\cdot)]^0 \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^n)$, by Corollary 2.149 in [22], there exists $\tilde{h}(\cdot,\cdot)$, $\tilde{\tilde{h}}(\cdot,\cdot) \in L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathbb{R}^n))$ such that for almost everywhere $\tau \in [t,T]$,

$$[p_1(\tau)]^0 = \mathbb{E}_t[p_1(\tau)]^0 + \int_t^\tau \tilde{h}(\tau,r)dW(r), \quad [k_1(\tau)]^0 = \mathbb{E}_t[k_1(\tau)]^0 + \int_t^\tau \tilde{\tilde{h}}(\tau,r)dW(r),$$

which and (4.7) yield that for almost everywhere $\theta \in [-\delta, 0]$,

$$[\mathbf{I}(t)](\theta) = \Bigg[\sum_{i=0}^{N}\Big(D_i^\top[k_1(t+\theta-\theta_i)]^0 + B_i^\top[p_1(t+\theta-\theta_i)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\theta_i)$$
$$+\int_{-\delta}^0\Big(D^0(\beta)^\top[k_1(t+\theta-\beta)]^0 + B^0(\beta)^\top[p_1(t+\theta-\beta)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\beta)d\beta\Bigg]$$
$$-\int_t^{T\wedge(t+\delta+\theta)}\Bigg[\sum_{i=0}^{N}\Big(B_i^\top\tilde{h}(t+\theta-\theta_i,r) + D_i^\top\tilde{\tilde{h}}(t+\theta-\theta_i,r)\Big)\mathbf{1}_{[t+\theta-T,t+\theta-r]}(\theta_i)$$
$$+\int_{-\delta}^0\Big(D^0(\beta)^\top\tilde{\tilde{h}}(t+\theta-\beta,r) + B^0(\beta)^\top\tilde{h}(t+\theta-\beta,r)\Big)\mathbf{1}_{[t+\theta-T,t+\theta-r]}(\beta)d\beta\Bigg]dW(r).$$

Define

$$[\tilde{k}(r)](\theta) := \sum_{i=0}^{N}\Big(D_i^\top\tilde{\tilde{h}}(r+\theta-\theta_i,r) + B_i^\top\tilde{h}(r+\theta-\theta_i,r)\Big)\mathbf{1}_{[r+\theta-T,\theta]}(\theta_i)$$
$$+\int_{-\delta}^0\Big(D^0(\beta)^\top\tilde{\tilde{h}}(r+\theta-\beta,r) + B(\beta)^\top\tilde{h}(r+\theta-\beta,r)\Big)\mathbf{1}_{[r+\theta-T,\theta]}(\beta)d\beta.$$

Then, by (4.4) and (4.5), we obtain

$$\mathbf{I}(t) = \int_t^T \mathcal{L}(r-t)^*\Big(\tilde{D}^*k_1(r) + \tilde{B}^*p_1(r)\Big)dr - \int_t^T \mathcal{L}(r-t)^*\tilde{k}(r)dW(r).$$

Let

$$\tilde{k}_2(r) := \int_r^T \mathcal{L}(\beta - r)^* h(\beta, r)d\beta + \tilde{k}(r).$$

Then, $(\tilde{p}_2(\cdot), \tilde{k}_2(\cdot))$ satisfies (4.6). Notice that $(p_1(\cdot), k_1(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([s,T];\mathfrak{M})) \times L^2_{\mathbb{F}}(s,T;\mathfrak{M})$, $h(\cdot,\cdot) \in L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathfrak{L}))$ and $\tilde{h}(\cdot,\cdot), \tilde{\tilde{h}}(\cdot,\cdot) \in L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathbb{R}^n))$. Then, we have $(\tilde{p}_2(\cdot), \tilde{k}_2(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([s,T];\mathfrak{L})) \times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$.

Step 2: Prove the necessity of Theorem 4.2.

Applying (3.3) and Theorem 3.3 in [10], we have

$$\mathbb{E}\int_s^T\Big(\langle p_1(t), \tilde{B}\mathbf{Y}_t^0\rangle_{\mathfrak{M}} + \langle k_1(t), \tilde{D}\mathbf{Y}_t^0\rangle_{\mathfrak{M}}\Big)dt$$
$$= \mathbb{E}\int_s^T\langle \mathbf{X}^0(t), \tilde{Q}(t)\bar{\mathbf{X}}(t) + \tilde{S}_1(t)^*\bar{\mathbf{Y}}_t + \tilde{S}_0(t)^*\bar{u}(t)\rangle_{\mathfrak{M}}dt + \mathbb{E}\langle \tilde{G}\bar{\mathbf{X}}(T), \mathbf{X}^0(T)\rangle_{\mathfrak{M}}.$$

Noting for any $f(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathfrak{L})$, we have

$$\mathbb{E}\int_s^T \big\langle \mathbf{Y}^0_t, f(t)\big\rangle_{L^2} dt = \mathbb{E}\int_s^T \Big\langle u^0(t), \int_t^{T\wedge(t+\delta)} [f(r)](t-r)dr\Big\rangle dt. \tag{4.8}$$

By $k_2(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathfrak{L})$, we deduce

$$\begin{aligned}
&\mathbb{E}\int_{-\delta}^0\int_{s+\theta}^s |[k_2(t-\theta)](\theta)|^2 dt d\theta + \mathbb{E}\int_{-\delta}^0\int_s^{T+\theta} |[k_2(t-\theta)](\theta)|^2 dt d\theta\\
&= \mathbb{E}\int_{-\delta}^0\int_{s+\theta}^{T+\theta} |[k_2(t-\theta)](\theta)|^2 dt d\theta = \mathbb{E}\int_s^T\int_{-\delta}^0 |[k_2(r)](\theta)|^2 d\theta dr < \infty,
\end{aligned}$$

which implies that

$$\mathbb{E}\int_t^{T\vee(t+\delta)} |[k_2(r)](t-r)|^2 dr < \infty, \quad \text{a.e. } t\in[s,T].$$

Thus, we obtain

$$\mathbb{E}\int_s^T \Big\langle u^0(t), \int_t^{T\wedge(t+\delta)} [k_2(r)](t-r)dW(r)\Big\rangle dt = \mathbb{E}\int_s^T \Big\langle u^0(t), \mathbb{E}_t\int_t^{T\wedge(t+\delta)} [k_2(r)](t-r)dW(r)\Big\rangle dt = 0. \tag{4.9}$$

By the definition of $\tilde{B}$, we derive

$$\begin{aligned}
&\mathbb{E}\int_s^T \big\langle p_1(t), \tilde{B}\mathbf{Y}^0_t\big\rangle_{\mathfrak{M}} dt = \mathbb{E}\int_s^T \Big\langle [p_1(t)]^0, \sum_{i=0}^N B_i u^0(t+\theta_i)\\
&\times \mathbf{1}_{(s-t,0]}(\theta_i) + \int_{-\delta}^0 B^0(\beta)u^0(t+\beta)\mathbf{1}_{(s-t,0]}(\beta)d\beta\Big\rangle dt\\
&= \mathbb{E}\int_s^T \Big\langle u^0(t), \int_{[-\delta,0]} B(d\beta)^\top [p_1(t-\beta)]^0 \mathbf{1}_{[t-T,0]}(\beta)\Big\rangle dt. \qquad (4.10)
\end{aligned}$$

By the definition of $\tilde{D}$, we obtain

$$\mathbb{E}\int_s^T \big\langle k_1(t), \tilde{D}\mathbf{Y}^0_t\big\rangle_{\mathfrak{M}} dt = \mathbb{E}\int_s^T \Big\langle u^0(t), \int_{[-\delta,0]} D(d\beta)^\top [k_1(t-\beta)]^0 \mathbf{1}_{[t-T,0]}(\beta)\Big\rangle dt. \tag{4.11}$$

By (4.8)–(4.11) and applying the convex variation technique in Theorem 4.1 in [29], we complete the proof of necessity.

Step 3: Prove the sufficiency of Theorem 4.2.

In fact, sufficiency is implied by the proof of necessity, thus we complete the proof. □

Remark 4.3. Since the new control operator $\mathbf{B}$ in (3.10) makes the transformed Problem (EP) not a standard infinite dimensional stochastic optimal control problem, a novel equation (4.2)(d) is introduced as an adjoint equation of (4.2)(b). For the deterministic system, the solvability of (4.2)(d) is natural, and does not need to be proved separately. While in the stochastic system, due to the backward structure, its solution contains two components $p_2(\cdot)$ and $k_2(\cdot)$, so an additional proof is required. From the above proof, for a.e. $\theta\in[-\delta,0]$, it is equivalent to the backward SEE (4.6) in $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^m)$, consisting of $\mathbb{R}^m$-valued $\mathcal{F}_T$-measurable random variables ξ such that $\mathbb{E}|\xi|^2<\infty$. Moreover, When $C_0, C^0(\theta), D_i, D^0(\theta)$ depend on t, Theorem 4.2 still holds, $i=0,\cdots,N$.

Inspired by (4.2) and (4.6), we go back to the original delayed control problem and characterize the open-loop solvability for Problem (P).

THEOREM 4.4. *Let* (A1)–(A2) *hold and* $G_{10}, G_{11}=0$. *For any given initial data* $(s,x,\varphi,\psi)\in[0,T)\times\mathfrak{Z}$, $\bar{u}(\cdot)$ *is an open-loop optimal control of* Problem (P) *if and only if the following two conditions hold:*

(i) (Stationarity condition)

$$\begin{aligned}
&\mathcal{M}(t) + S_{00}(t)\bar{X}(t) + R_{00}(t)\bar{u}(t)\\
&+\int_{-\delta}^0\Big[R_{10}(t,\theta)^\top\bar{u}(t+\theta) + S_{01}(t,\theta)\bar{X}(t+\theta)\Big]d\theta = 0, \ \text{a.e. a.s.}, \qquad (4.12)
\end{aligned}$$

where

$$\mathcal{M}(t):=\mathbb{E}_t\bigg[\int_t^{T\wedge(t+\delta)}\Big(S_{10}(r,t-r)\bar{X}(r)+R_{10}(r,t-r)\bar{u}(r)+\int_{-\delta}^{0}\big[R_{11}(r,\theta,t-r)\bar{u}(r+\theta)+S_{11}(r,\theta,t-r)\bar{X}(r+\theta)\big]d\theta+B^0(t-r)^\top\mathfrak{P}(r)+D^0(t-r)^\top\mathfrak{Q}(r)\Big)dr+\mathbf{1}_{[0,T+\theta_i]}(t)\sum_{i=0}^{N}\Big(B_i^\top\mathfrak{P}(t-\theta_i)+D_i^\top\mathfrak{Q}(t-\theta_i)\Big)\bigg],$$

with $(\bar{X}(\cdot),\mathfrak{P}(\cdot),\mathfrak{Q}(\cdot))$ *satisfying the following anticipated-backward SDDE:*

$$(4.13)\quad \begin{cases} d\bar{X}(t)=\int_{[-\delta,0]}\Big(A(d\theta)\bar{X}_t(\theta)+B(d\theta)\bar{u}_t(\theta)\Big)dt\\ \qquad+\int_{[-\delta,0]}\Big(C(d\theta)\bar{X}_t(\theta)+D(d\theta)\bar{u}_t(\theta)\Big)dW(t),\quad t\in[s,T],\\ -d\mathfrak{P}(t)=\Big\{\sum_{i=0}^{N}A_i^\top\mathbb{E}_t[\mathfrak{P}(t-\theta_i)]\mathbf{1}_{[0,T+\theta_i)}(t)+C_0(t)^\top\mathfrak{Q}(t)+Q_{00}(t)\bar{X}(t)\\ \qquad+S_{00}(t)^\top\bar{u}(t)+\int_{-\delta}^{0}\Big(S_{10}(t,\theta)^\top\bar{u}(t+\theta)+Q_{10}(t,\theta)^\top\bar{X}(t+\theta)\Big)d\theta\\ \qquad+\int_{(t-T)\vee(-\delta)}^{0}\mathbb{E}_t\Big(A^0(\theta)^\top\mathfrak{P}(t-\theta)+C^0(\theta)^\top\mathfrak{Q}(t-\theta)+Q_{10}(t-\theta,\theta)\\ \qquad\times\bar{X}(t-\theta)+S_{01}(t-\theta,\theta)^\top\bar{u}(t-\theta)+\int_{-\delta}^{0}[Q_{11}(t-\theta,\theta',\theta)\bar{X}(t-\theta+\theta')\\ \qquad+S_{11}(t-\theta,\theta,\theta')^\top\bar{u}(t-\theta+\theta')]d\theta'\Big)d\theta\Big\}dt-\mathfrak{Q}(t)dW(t),\ t\in[s,T],\\ \bar{X}(s)=x,\ \bar{X}(t)=\varphi(t-s),\ t\in[s-\delta,s),\ \bar{u}(t)=\psi(t-s),\ t\in[s-\delta,s],\\ \mathfrak{P}(T)=G_{00}\bar{X}(T). \end{cases}$$

(ii) (Convexity condition)

$$J(s,0,0,0;u^0(\cdot))\geqslant 0,\ \forall u^0(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m),$$

where $X^0(\cdot)$ *satisfies the following SDDE:*

$$\begin{cases} dX^0(t)=\int_{[-\delta,0]}\Big(A(d\theta)X^0_t(\theta)+B(d\theta)u^0_t(\theta)\Big)dt\\ \qquad+\int_{[-\delta,0]}\Big(C(d\theta)X^0_t(\theta)+D(d\theta)u^0_t(\theta)\Big)dW(t),\quad t\in[s,T],\\ X^0(t)=0,\ u^0(t)=0,\quad t\in[s-\delta,s]. \end{cases}$$

Proof. Using the convex variational technique and applying Itô formula to $\langle\mathfrak{P}(\cdot),X^0(\cdot)\rangle$, the proof is completed, similar to the proof of Theorem 4.1 in [29]. ∎

Remark 4.5. (i) From (4.1) and (4.12), an interesting thing is that if $[p_1(t)]^0=\mathfrak{P}(t)$, $[k_1(t)]^0=\mathfrak{Q}(t)$ for all $t\in[s,T]$, then $\mathcal{M}(t)=\mathbb{E}_t([p_2(t)](0))=[p_2(t)](0)$, thus the stationarity conditions (4.1) and (4.12) are consistent. (ii) Theorem 4.4 is derived similarly, when the coefficients of the state equation (2.3) are time-variant. (iii) Let delay disappear in Problem (P). Then, Theorem 4.4 reduces to Theorem 2.3.2 in [31] when $b,\sigma,g,q,\rho=0$ there. (iv) Let Problem (P) only contain pointwise delay and A_i, B_i,$D_i=0, i=1,\cdots,N-1$. Then, the second equation of (4.13) is similar to (12) in [4].

5. Closed-loop representation of open-loop optimal control. In this section, we study the solvability of an integral operator-valued Riccati equation, inspired by which, we give the closed-loop representation of the open-loop optimal control for Problem (P), by introducing a coupled matrix-valued Riccati equation.

Definition 5.1. *An open-loop optimal control* $\bar{u}(\cdot)$ *of* Problem (P) *is said to admit a* closed-loop representation*, if there exists* $\bar{K}(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ *such that for any initial data* $(x,\varphi,\psi)\in\mathfrak{Z}$*, the function*

$$\bar{u}(t) := \bar{K}(t)\bar{\mathbf{Z}}(t), \quad t \in [s,T],$$

is an open-loop optimal control of Problem (P) *for* $(x,\varphi,\psi) \in \mathfrak{Z}$, *where* $\bar{\mathbf{Z}}(\cdot)$ *is the solution to the following closed-loop system with* $\mathbf{Z}_0 := (x^\top, \varphi^\top, \psi^\top)^\top$*:*

$$\bar{\mathbf{Z}}(t) = \mathbf{T}(t-s)\mathbf{Z}_0 + \int_s^t \mathbf{T}(t-r)\mathbf{B}\bar{K}(r)\bar{\mathbf{Z}}(r)dr + \int_s^t \mathbf{T}(t-r)\mathbf{C}\bar{\mathbf{Z}}(r)dW(r), t \in [s,T]. \tag{5.1}$$

For any $z \in \mathfrak{Z}$, consider the following integral operator-valued Riccati equation:

$$\begin{aligned} P(t)z = \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z + \int_t^T \mathbf{T}(r-t)^*\Big[\mathbf{C}^*P(r)\mathbf{C} + \mathbf{Q}(r) \\ -(\mathbf{B}^*P(r))^*\mathbf{R}(r)^{-1}(\mathbf{B}^*P(r))\Big]\mathbf{T}(r-t)zdr, \quad t \in [s,T], \end{aligned} \tag{5.2}$$

where $\mathbf{B}^* := (0, \Delta^*)$. The following theorem guarantees its solvability.

THEOREM 5.2. *Suppose all coefficients of* Problem (P) *are continuous and* $\mathbf{C} \in \mathscr{L}(\mathfrak{Z})$. *Assume that there exists a constant* $\mu > 0$ *such that* $R_{00} \geqslant \mu$. *Then, the integral operator-valued Riccati equation* (5.2) *admits a unique solution in the class of strongly continuous self-adjoint operators.*

Proof. In the following, denote $||\cdot||_{\mathscr{L}(\mathfrak{Z})}$, $||\cdot||_{\mathfrak{Z}}$ by $||\cdot||$ for simplicity. First we show that there exists $T_0 \in [0, T-s]$, such that (5.2) admits a unique solution on $[T-T_0, T]$.

Let $\mathcal{B}(l) := \Big\{ P(\cdot) : [T-T_0, T] \to \mathscr{L}(\mathfrak{Z}) \Big| P(\cdot)$ is a strongly continuous self-adjoint operator, $\sup_{t\in[T-T_0,T]} ||P(t)|| \leqslant l \Big\}$. Consider the mapping: $\mathscr{T}: \mathcal{B}(l) \to \mathcal{B}(l)$, $\tilde{P}(\cdot) \mapsto P(\cdot)$, and $P(\cdot) = \mathscr{T}(\tilde{P}(\cdot))$ satisfies the following integral equation, for any $z \in \mathfrak{Z}$, $t \in [T-T_0, T]$,

$$\begin{aligned} P(t)z = \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z + \int_t^T \mathbf{T}(r-t)^*\big[\mathbf{C}^*\tilde{P}(r)\mathbf{C} + \mathbf{Q}(r) \\ -(\mathbf{B}^*P(r))^*\mathbf{R}(r)^{-1}(\mathbf{B}^*P(r))\big]\mathbf{T}(r-t)zdr. \end{aligned} \tag{5.3}$$

Then, we'll complete the proof of this part in two steps.

Step 1: Show that $\mathscr{T}$ is well-defined.

Define $\tau := T - T_0$, consider the following optimal control problem:

$$\begin{cases} \tilde{Z}(t) = \mathbf{T}(t-\tau)z_0 + \int_\tau^t \mathbf{T}(t-r)\mathbf{B}u(r)dr, \quad t \in [\tau, T],\ z_0 \in \mathfrak{Z}, \\ \min_{u(\cdot)\in L^2(\tau,T;\mathbb{R}^m)} \tilde{J}(\tau, z_0; u(\cdot)) = \int_\tau^T \Big[\Big\langle \Big(\mathbf{C}^*\tilde{P}(t)\mathbf{C} + \mathbf{Q}(t)\Big)\tilde{Z}(t), \tilde{Z}(t)\Big\rangle + \Big\langle \mathbf{R}(t)u(t), u(t)\Big\rangle\Big]dt \\ \qquad + \langle \mathbf{G}\tilde{Z}(T), \tilde{Z}(T)\rangle. \end{cases}$$

Then, similar to Theorem 2.3 in [12], the optimal control is $\bar{\tilde{u}}(t) = -\mathbf{R}(t)^{-1}\mathbf{B}^*P(t)\bar{\tilde{Z}}(t)$, and the value function is $\tilde{V}(\tau, z_0) = \big\langle P(\tau)z_0, z_0\big\rangle$, where $P(\cdot)$ satisfies (5.3). Moreover, similar to Lemma 2.6 in [12], (5.3) is equivalent to the following equation:

$$\begin{cases} P(t)z = \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}_\infty(T,t)z + \int_t^T \mathbf{T}(r-t)^*\Big(\mathbf{C}^*\tilde{P}(r)\mathbf{C} + \mathbf{Q}(r)\Big)\mathbf{T}_\infty(r,t)zdr, \\ \mathbf{T}_\infty(r,t)z = \mathbf{T}(r-t)z - \int_t^r \mathbf{T}(r-\beta)\mathbf{B}\mathbf{R}(\beta)^{-1}\mathbf{B}^*P(\beta)\mathbf{T}_\infty(\beta,t)zd\beta, \tau \leqslant t \leqslant r \leqslant T. \end{cases} \tag{5.4}$$

Let $P_0(\cdot)$ be the solution to the following integral equation:

$$P_0(t)z = \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z + \int_t^T \mathbf{T}(r-t)^*\Big(\mathbf{C}^*\tilde{P}(r)\mathbf{C} + \mathbf{Q}(r)\Big)\mathbf{T}(r-t)zdr, t \in [\tau, T].$$

Then, we have $P(t) \leqslant P_0(t)$. Thus, we obtain

$$||P(t)|| \leqslant ||\mathbf{G}||\gamma'^2(e^{2\gamma T_0} \vee 1) + \gamma'^2 T_0(e^{2\gamma T_0} \vee 1)\Big(\sup_{s\leqslant r\leqslant T} ||\mathbf{Q}(r)|| + l||\mathbf{C}||^2\Big), t \in [\tau, T], \tag{5.5}$$

where $\gamma' \geqslant 1$ and $\gamma \in \mathbb{R}$ satisfying that $||\mathbf{T}(t)|| \leqslant \gamma' e^{\gamma t}$ for all $t \in [s,T]$. Choose large enough l and small enough T_0 such that $\gamma'^2 T_0(e^{2\gamma T_0} \vee 1)||\mathbf{C}||^2 < \frac{1}{2}$, and

$$l > 2\gamma'^2(e^{2\gamma T_0} \vee 1)(T_0+2)\big(||\mathbf{G}|| + \sup_{s\leqslant r\leqslant T}||\mathbf{Q}(r)||\big).$$

Then, we have $\sup_{\tau\leqslant t\leqslant T}||P(t)|| < l$, thus $\mathscr{T}$ is well-defined.

Step 2: Show that $\mathscr{T}$ is a contraction mapping.

Denote $\hat{\tilde{P}}(\cdot) = \tilde{P}_1(\cdot) - \tilde{P}_2(\cdot)$, $\hat{P}(\cdot) = P_1(\cdot) - P_2(\cdot)$, and $\hat{\mathbf{T}}_\infty(\cdot,\cdot) = \mathbf{T}^1_\infty(\cdot,\cdot) - \mathbf{T}^2_\infty(\cdot,\cdot)$. Then, we get

$$||\mathbf{T}_\infty(r,t)|| \leqslant M(T_0), \quad \tau \leqslant t \leqslant r \leqslant T.$$

Here and after, $M(T_0)$ is a generic constant, depending on $\mu, T_0, |B_i|, \sup_{\theta\in[-\delta,0]}|B^0(\theta)|, ||\mathbf{G}||, \sup_{s\leqslant r\leqslant T}||\mathbf{Q}(r)||, ||\Phi||, ||\mathcal{L}||, ||\mathbf{C}||, l$. And $M(T_0)$ increases as T_0 increases. By (5.4) we have

$$\sup_{\tau\leqslant t\leqslant T}||\hat{\mathbf{T}}_\infty(t,\tau)||^2 \leqslant M(T_0)\int_\tau^T||\hat{P}(r)||^2dr, \quad \sup_{\tau\leqslant t\leqslant T}||\hat{P}(t)||^2 \leqslant M(T_0)\sup_{\tau\leqslant t\leqslant T}||\hat{\tilde{P}}(t)||^2.$$

Choose T_0 such that $M(T_0)$ in the above inequality satisfies

$$M(T_0) < 1. \tag{5.6}$$

Then, $\mathscr{T}$ is a contraction mapping on $[T-T_0,T]$, thus there exists T_0 such that (5.2) admits a unique solution on $[T-T_0,T]$.

Finally, we aim to show that (5.2) admits a unique solution on the whole interval $[s,T]$.

For any $z \in \mathfrak{Z}$ and $t \in [T-T_0,T]$, consider

$$\bar{P}_0(t)z = \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z + \int_t^T \mathbf{T}(r-t)^*\Big(\mathbf{C}^*\bar{P}_0(r)\mathbf{C} + \mathbf{Q}(r)\Big)\mathbf{T}(r-t)zdr.$$

Then, for $t \in [T-T_0,T]$, $||P(t)|| \leqslant ||\bar{P}_0(t)|| \leqslant \tilde{l}$, where $\tilde{l}$ depends on $|B_i|, \sup_{\theta\in[-\delta,0]}|B^0(\theta)|, \sup_{r\in[s,T]}||\mathbf{Q}(r)||, ||\mathbf{G}||, ||\Phi||, ||\mathcal{L}||, T_0, ||\mathbf{C}||$. On $[T-T_0-T_1, T-T_0]$, consider the mapping $\mathscr{T}$, in this case, $\mathbf{G}$ is replaced by $P(T-T_0)$ in the above part. Choose small enough T_1 and large enough l such that $\gamma'^2T_1(e^{2\gamma T_1} \vee 1)\|\mathbf{C}\|^2 < \frac{1}{2}$ and $l > 2\gamma'^2(e^{2\gamma T_1} \vee 1)(T_1 + 2)(\tilde{l} + \sup_{r\in[s,T]}||\mathbf{Q}(r)||)$. Then, similar to (5.5), we get for $t \in [T-T_0-T_1, T-T_0]$,

$$||P(t)|| \leqslant ||P(T-T_0)||(e^{2\gamma T_1} \vee 1)\gamma'^2 + \gamma'^2T_1(e^{2\gamma T_1} \vee 1)\Big(\sup_{s\leqslant r\leqslant T}||\mathbf{Q}(r)|| + l||\mathbf{C}||^2\Big) < l,$$

thus $\mathscr{T}$ is well-defined on $[T-T_1-T_0, T-T_0]$. Similar to (5.6), let $M(T_1) < 1$. Then, $\mathscr{T}$ is a contraction mapping on $[T-T_1-T_0, T-T_0]$. Repeating the above steps, (5.2) admits a unique solution on $[s,T]$, which completes the proof of Theorem 5.2. ∎

In the rest of this section, we consider Problem (P) with the following state equation instead of (2.3):

$$\begin{cases} dX(t) = \Big[\sum_{i=0}^N A_iX(t+\theta_i) + \int_{-\delta}^0 A^0(\theta)X(t+\theta)d\theta + \sum_{i=0}^N B_iu(t+\theta_i) \\ \qquad + \int_{-\delta}^0 B^0(\theta)u(t+\theta)d\theta\Big]dt + \Big[C_0X(t) + \int_{-\delta}^0 C^0(\theta)X(t+\theta)d\theta \\ \qquad + D_0u(t) + \int_{-\delta}^0 D^0(\theta)u(t+\theta)d\theta\Big]dW(t), \quad t \in [s,T], \\ X(s) = x, X(t) = \varphi(t-s), t \in [s-\delta, s), u(t) = \psi(t-s), t \in [s-\delta, s]. \end{cases} \tag{5.7}$$

Inspired by (5.2), let $P_{00}(t)\xi = \Big[P(t)\binom{\xi}{0}\Big]^0$, $P_{01}(t)\psi = \Big[P(t)\binom{0}{\psi}\Big]^0$, $P_{10}(t)\xi = \Big[P(t)\binom{\xi}{0}\Big]^1$, $P_{11}(t)\psi = \Big[P(t)\binom{0}{\psi}\Big]^1$. Then, under some proper conditions on the coefficients, $(P_{00}(\cdot), P_{01}(\cdot), P_{10}(\cdot), P_{11}(\cdot))$ satisfies the following differential operator-valued Riccati equation:

$$
(5.8)\quad \begin{cases}
(a)\,\dot P_{00}(t)=-\tilde A^*P_{00}(t)-P_{00}(t)\tilde A-\tilde C^*P_{00}(t)\tilde C-\mathbf{Q}(t)+\big(\Delta^*P_{10}(t)\big)^*R_{00}(t)^{-1}\big(\Delta^*P_{10}(t)\big),\\
(b)\,\dot P_{01}(t)=-\tilde A^*P_{01}(t)-P_{00}(t)\tilde B-P_{01}(t)\mathcal{A}-\tilde C^*P_{00}(t)\tilde D+\big(\Delta^*P_{10}(t)\big)^*R_{00}(t)^{-1}\big(\Delta^*P_{11}(t)\big),\\
(c)\,\dot P_{10}(t)=-\tilde B^*P_{00}(t)-\mathcal{A}^*P_{10}(t)-P_{10}(t)\tilde A-\tilde D^*P_{00}(t)\tilde C\\
\qquad\qquad +\big(\Delta^*P_{11}(t)\big)^*R_{00}(t)^{-1}\big(\Delta^*P_{10}(t)\big),\\
(d)\,\dot P_{11}(t)=-\tilde B^*P_{01}(t)-\mathcal{A}^*P_{11}(t)-P_{10}(t)\tilde B-P_{11}(t)\mathcal{A}-\tilde D^*P_{00}(t)\tilde D\\
\qquad\qquad +\big(\Delta^*P_{11}(t)\big)^*R_{00}(t)^{-1}\big(\Delta^*P_{11}(t)\big),\\
\quad P_{00}(T)=\tilde G,\, P_{01}(T)=0,\, P_{10}(T)=0,\, P_{11}(T)=0.
\end{cases}
$$

Next we decompose (5.8), adjust some terms in the equations for $P_{00}(\cdot)$, $P_{01}(\cdot)$, $P_{11}(\cdot)$, and introduce the following Riccati equations. Denote $\mathfrak{R}(t) := R_{00}(t) + {D_0}^\top E_0(t)D_0$. Then, inspired by (5.8)(a), for almost everywhere $t\in[s,T], \theta,\alpha\in[-\delta,0]$, introduce the coupled matrix-valued Riccati equation:

$$
(5.9)\quad \begin{cases}
\dot E_0(t)=-A_0^\top E_0(t)-E_0(t)A_0-E_1(t,0)-E_1(t,0)^\top-C_0^\top E_0(t)C_0\\
-Q_{00}(t)+\Big(E_3(t,0)+S_{00}(t)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0\Big)^\top\\
\times\mathfrak{R}(t)^{-1}\Big(E_3(t,0)+S_{00}(t)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0\Big),\\
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}\Big)E_1(t,\theta)=-E_1(t,\theta)A_0-E_2(t,\theta,0)-\Big[\displaystyle\sum_{i=1}^{N-1}A_i\hat\delta(\theta-\theta_i)+A^0(\theta)\Big]^\top E_0(t)\\
-Q_{10}(t,\theta)-C^0(\theta)^\top E_0(t)C_0+\Big[E_4(t,0,\theta)+S_{01}(t,\theta)+B_0^\top E_1(t,\theta)^\top\\
+D_0^\top E_0(t)C^0(\theta)\Big]^\top\mathfrak{R}(t)^{-1}\Big[E_3(t,0)+S_{00}(t)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0\Big],\\
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}-\dfrac{\partial}{\partial\alpha}\Big)E_2(t,\theta,\alpha)=-\Big[A^0(\theta)+\displaystyle\sum_{i=1}^{N-1}A_i\hat\delta(\theta-\theta_i)\Big]^\top E_1(t,\alpha)^\top-E_1(t,\theta)\Big[A^0(\alpha)\\
+\displaystyle\sum_{i=1}^{N-1}A_i\hat\delta(\alpha-\theta_i)\Big]-C^0(\theta)^\top E_0(t)C^0(\alpha)-Q_{11}(t,\alpha,\theta)+\Big[E_4(t,0,\theta)+S_{01}(t,\theta)+B_0^\top E_1(t,\theta)^\top\\
+D_0^\top E_0(t)C^0(\theta)\Big]^\top\mathfrak{R}(t)^{-1}\Big[E_4(t,0,\alpha)+S_{01}(t,\alpha)+B_0^\top E_1(t,\alpha)^\top+{D_0}^\top E_0(t)C^0(\alpha)\Big],\\
\quad E_0(T)=G_{00},\ E_1(T,\theta)=G_{10}(\theta),\ E_1(t,-\delta)=A_N^\top E_0(t),\\
\quad E_2(T,\theta,\alpha)=G_{11}(\alpha,\theta),\, E_2(t,-\delta,\alpha)=A_N^\top E_1(t,\alpha)^\top,\, E_2(t,\theta,-\delta)=E_1(t,\theta)A_N.
\end{cases}
$$

Similarly, inspired by (5.8)(b), introduce the coupled matrix-valued Riccati equation:

$$
(5.10)\quad \begin{cases}
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}\Big)E_3(t,\theta)=-\Big[\displaystyle\sum_{i=1}^{N-1}B_i\hat\delta(\theta-\theta_i)+B^0(\theta)\Big]^\top E_0(t)-D^0(\theta)^\top E_0(t)C_0-E_4(t,\theta,0)\\
-S_{10}(t,\theta)+\Big[E_5(t,0,\theta)+R_{10}(t,\theta)^\top+B_0^\top E_3(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\Big]^\top\mathfrak{R}(t)^{-1}\\
\times\Big[E_3(t,0)+S_{00}(t)+B_0^\top E_0(t)^\top+D_0^\top E_0(t)C_0\Big]-E_3(t,\theta)A_0,\\
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}-\dfrac{\partial}{\partial\alpha}\Big)E_4(t,\theta,\alpha)=-\Big[B^0(\theta)+\displaystyle\sum_{i=1}^{N-1}B_i\hat\delta(\theta-\theta_i)\Big]^\top E_1(t,\alpha)^\top-E_3(t,\theta)\Big[A^0(\alpha)\\
+\displaystyle\sum_{i=1}^{N-1}A_i\hat\delta(\alpha-\theta_i)\Big]-D^0(\theta)^\top E_0(t)C^0(\alpha)-S_{11}(t,\alpha,\theta)+\Big[E_5(t,0,\theta)+R_{10}(t,\theta)^\top+B_0^\top E_3(t,\theta)^\top\\
+D_0^\top E_0(t)D^0(\theta)\Big]^\top\mathfrak{R}(t)^{-1}\Big[E_4(t,0,\alpha)+S_{01}(t,\alpha)+B_0^\top E_1(t,\alpha)^\top+{D_0}^\top E_0(t)C^0(\alpha)\Big],\\
E_3(T,\theta)=0,\ E_3(t,-\delta)=B_N^\top E_0(t),\\
E_4(T,\theta,\alpha)=0,\ E_4(t,-\delta,\alpha)=B_N^\top E_1(t,\alpha)^\top,\ E_4(t,\theta,-\delta)=E_3(t,\theta)A_N.
\end{cases}
$$

Inspired by (5.8)(d), introduce the following matrix-valued Riccati equation:

$$(5.11)\quad \begin{cases} \Big(\frac{\partial}{\partial t}-\frac{\partial}{\partial\theta}-\frac{\partial}{\partial\alpha}\Big)E_5(t,\theta,\alpha)=-\Big[\sum_{i=1}^{N-1}B_i\hat{\delta}(\theta-\theta_i)+B^0(\theta)\Big]^\top E_3(t,\alpha)^\top \\ \quad -E_3(t,\theta)\Big[\sum_{i=1}^{N-1}B_i\hat{\delta}(\alpha-\theta_i)+B^0(\alpha)\Big]-D^0(\theta)^\top E_0(t)D^0(\alpha)-R_{11}(t,\alpha,\theta) \\ \quad +\Big[E_5(t,0,\theta)+D_0^\top E_0(t)D^0(\theta)+R_{10}(t,\theta)^\top+B_0^\top E_3(t,\theta)^\top\Big]^\top \\ \quad \times\mathfrak{R}(t)^{-1}\Big[E_5(t,0,\alpha)+D_0^\top E_0(t)D^0(\alpha)+R_{10}(t,\alpha)^\top+B_0^\top E_3(t,\alpha)^\top\Big],\ \text{a.e. } t,\alpha,\theta, \\ E_5(T,\theta,\alpha)=0,\ E_5(t,-\delta,\alpha)=B_N^\top E_3(t,\alpha)^\top,\ E_5(t,\theta,-\delta)=E_3(t,\theta)B_N, \end{cases}$$

where $\hat{\delta}(\cdot)$ is the delta function, i.e. $\hat{\delta}(\theta)=0$ for $\theta\neq 0$ and $\int_{-\infty}^{\infty}\hat{\delta}(\theta)d\theta=1$. Then, we can derive the closed-loop representation of open-loop optimal control for Problem (P).

THEOREM 5.3. *Suppose all coefficients of* Problem (P) *are continuous and* $\mathfrak{R}>0$. *Let continuous functions* $E_0(t)$, $E_1(t,\theta)$, $E_2(t,\theta,\alpha)$, $E_3(t,\theta)$, $E_4(t,\theta,\alpha)$, $E_5(t,\theta,\alpha)$, $t\in[s,T]$, $\theta,\alpha\in[-\delta,0]$, *satisfy the coupled matrix-valued Riccati equations* (5.9)–(5.11), *and* $E_0(t)=E_0(t)^\top$, $E_2(t,\theta,\alpha)=E_2(t,\alpha,\theta)^\top$, $E_5(t,\theta,\alpha)=E_5(t,\alpha,\theta)^\top$. *For any given initial data* $(s,x,\varphi,\psi)\in[0,T)\times\mathfrak{Z}$, *denote*

$$\bar{K}(t)\begin{pmatrix}x\\ \varphi\\ \psi\end{pmatrix}=-\mathfrak{R}(t)^{-1}\Big\{\Big[E_3(t,0)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0+S_{00}(t)\Big]x$$
$$+\int_{-\delta}^0\Big[E_4(t,0,\theta)+B_0^\top E_1(t,\theta)^\top+S_{01}(t,\theta)+D_0^\top E_0(t)C^0(\theta)\Big]\varphi(\theta)d\theta$$
$$(5.12)\quad +\int_{-\delta}^0\Big[E_5(t,0,\theta)+B_0^\top E_3(t,\theta)^\top+R_{10}(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\Big]\psi(\theta)d\theta\Big\}.$$

Then, the closed-loop representation of the open-loop optimal control for Problem (P) *with the state equation* (5.7), *is as follows:*

$$(5.13)\qquad \bar{u}(t)=\bar{K}(t)\bar{\mathbf{Z}}(t),\quad \text{a.e. a.s.},$$

where $\bar{\mathbf{Z}}(\cdot)$ *satisfies* (5.1), *and the value function has the following form:*

$$\begin{aligned}V(s,x,\varphi(\cdot),\psi(\cdot))&=\langle E_0(s)x,x\rangle+2\int_{-\delta}^0\langle\varphi(\theta),E_1(s,\theta)x\rangle d\theta\\ &+\int_{-\delta}^0\int_{-\delta}^0\langle E_2(s,\theta,\alpha)\varphi(\alpha),\varphi(\theta)\rangle d\theta d\alpha+2\int_{-\delta}^0\langle\psi(\theta),E_3(s,\theta)x\rangle d\theta\\ &+2\int_{-\delta}^0\int_{-\delta}^0\langle\psi(\theta),E_4(s,\theta,\alpha)\varphi(\alpha)\rangle d\alpha d\theta+\int_{-\delta}^0\int_{-\delta}^0\langle E_5(s,\theta,\alpha)\psi(\alpha),\psi(\theta)\rangle d\theta d\alpha.\end{aligned}$$

Proof. Problem (P) is equivalent to Problem (EP) as noted in Remark 3.3, thus

$$\bar{u}(t)=-\mathfrak{R}(t)^{-1}\big\{\big[E_3(t,0)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0+S_{00}(t)\big]\bar{X}(t)$$
$$+\int_{-\delta}^0\big[E_4(t,0,\theta)+B_0^\top E_1(t,\theta)^\top+S_{01}(t,\theta)+D_0^\top E_0(t)C^0(\theta)\big]\bar{X}(t+\theta)d\theta$$
$$(5.14)\quad +\int_{-\delta}^0\big[E_5(t,0,\theta)+B_0^\top E_3(t,\theta)^\top+R_{10}(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\big]\bar{u}(t+\theta)d\theta\big\},\ \text{a.e. a.s.}$$

Then, by (2.4), we only need to prove that

$$J(s,x,\varphi(\cdot),\psi(\cdot);\bar{u}(\cdot))\leqslant J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot)),\quad\forall u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m).$$

Define

$$\begin{aligned}\Gamma(t)&:=\big\langle E_0(t)X(t),X(t)\big\rangle+2\int_{-\delta}^0\big\langle X(t+\theta),E_1(t,\theta)X(t)\big\rangle d\theta\\ &+\int_{-\delta}^0\int_{-\delta}^0\big\langle E_2(t,\theta,\alpha)X(t+\alpha),X(t+\theta)\big\rangle d\theta d\alpha+2\int_{-\delta}^0\big\langle u(t+\theta),E_3(t,\theta)X(t)\big\rangle d\theta\end{aligned}$$

$$+2\int_{-\delta}^{0}\int_{-\delta}^{0}\big\langle u(t+\theta),E_4(t,\theta,\alpha)X(t+\alpha)\big\rangle d\alpha d\theta+\int_{-\delta}^{0}\int_{-\delta}^{0}\big\langle E_5(t,\theta,\alpha)u(t+\alpha),u(t+\theta)\big\rangle d\theta d\alpha.$$

Then, by (5.7), (5.9)–(5.11) and applying Itô formula, we obtain

$$\begin{aligned}
&J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot))\\
&=\Gamma(s)+\mathbb{E}\int_s^T\Big\langle\mathfrak{R}(t)\Big(u(t)+\mathfrak{R}(t)^{-1}\Big\{\big[E_3(t,0)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0+S_{00}(t)\big]X(t)\\
&\quad+\int_{-\delta}^{0}\big[E_4(t,0,\theta)+B_0^\top E_1(t,\theta)^\top+S_{01}(t,\theta)+D_0^\top E_0(t)C^0(\theta)\big]X(t+\theta)d\theta\\
&\quad+\int_{-\delta}^{0}\big[E_5(t,0,\theta)+B_0^\top E_3(t,\theta)^\top+R_{10}(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\big]u(t+\theta)d\theta\Big\}\Big),\\
&\qquad u(t)+\mathfrak{R}(t)^{-1}\Big\{\big[E_3(t,0)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0+S_{00}(t)\big]X(t)\\
&\quad+\int_{-\delta}^{0}\big[E_4(t,0,\theta)+B_0^\top E_1(t,\theta)^\top+S_{01}(t,\theta)+D_0^\top E_0(t)C^0(\theta)\big]X(t+\theta)d\theta\\
&\quad+\int_{-\delta}^{0}\big[E_5(t,0,\theta)+B_0^\top E_3(t,\theta)^\top+R_{10}(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\big]u(t+\theta)d\theta\Big\}\Big\rangle dt,
\end{aligned}$$

which completes the proof. ∎

Remark 5.4. Now we study the solvability of the coupled matrix-valued Riccati equations (5.9)–(5.11). Assume that $A_i, B_i = 0$, $i = 1,\cdots,N-1$, and D_0, G_{00}, G_{10}, $G_{11} = 0$. Then, (5.9)–(5.11) admit unique solutions. Here we just provide a sketch of the proof, and we refer to [1] for full details of each step.

Step 1: Consider the integral forms of the coupled matrix-valued Riccati equations (5.9)–(5.11). Then, there exists $\tau > 0$ such that (5.9)–(5.11) admit unique solutions for $T-\tau \leqslant t \leqslant T$, $-\delta \leqslant \theta,\alpha \leqslant 0$. In fact, denote by M the upper bound of all coefficients of Problem (P), and for any given $l > 0$, define

$$\begin{aligned}
&\mathcal{B}(l) := \big\{(E_0(\cdot),E_1(\cdot,\cdot),E_2(\cdot,\cdot,\cdot),E_3(\cdot,\cdot),E_4(\cdot,\cdot,\cdot),E_5(\cdot,\cdot,\cdot))\in C([T-\tau,T];\mathbb{S}^n)\\
&\times C([T-\tau,T]\times[-\delta,0];\mathbb{R}^{n\times n})\times C([T-\tau]\times[-\delta,0]^2;\mathbb{R}^{n\times n})\times C([T-\tau,T]\times[-\delta,0];\mathbb{R}^{m\times n})\\
&\times C([T-\tau,T]\times[-\delta,0]^2;\mathbb{R}^{m\times n})\times C([T-\tau,T]\times[-\delta,0]^2;\mathbb{R}^{m\times m});\\
&\sup_{\substack{t\in[T-\tau,T]\\ \theta,\alpha\in[-\delta,0]}}\big\{|E_0(t)|+|E_1(t,\theta)|+|E_2(t,\theta,\alpha)|+|E_3(t,\theta)|+|E_4(t,\theta,\alpha)|+|E_5(t,\theta,\alpha)|\big\}\leqslant l\big\}.
\end{aligned}$$

Consider the mapping $\mathscr{T}:\mathcal{B}(l)\longrightarrow\mathcal{B}(l)$, $(E_0(\cdot),E_1(\cdot,\cdot),E_2(\cdot,\cdot,\cdot),E_3(\cdot,\cdot),E_4(\cdot,\cdot,\cdot),E_5(\cdot,\cdot,\cdot))$ $\mapsto(\tilde E_0(\cdot),\tilde E_1(\cdot,\cdot),\tilde E_2(\cdot,\cdot,\cdot),\tilde E_3(\cdot,\cdot),\tilde E_4(\cdot,\cdot,\cdot),\tilde E_5(\cdot,\cdot,\cdot))$, where $\tilde E_0(\cdot)$, $\tilde E_1(\cdot,\cdot)$ and $\tilde E_2(\cdot,\cdot,\cdot)$ satisfy the integral form of (5.9):

$$\begin{aligned}
\tilde E_0(t)=\int_t^T\Big[&A_0^\top E_0(s)+E_0(s)A_0+E_1(s,0)+E_1(s,0)^\top+C_0^\top E_0(s)C_0+Q_{00}(s)\\
&-\big(E_3(s,0)+S_{00}(s)+B_0^\top E_0(s)\big)^\top R_{00}(s)^{-1}\big(E_3(s,0)+S_{00}(s)+B_0^\top E_0(s)\big)\Big]ds,
\end{aligned}\tag{5.15}$$

$$\begin{aligned}
\tilde E_1(t,\theta)=A_N^\top\tilde E_0(t+\theta+\delta)\mathbf{1}_{[-\delta,T-t-\delta)}(\theta)+\int_t^{(t+\theta+\delta)\wedge T}&\Big\{-E_1(r,t+\theta-r)A_0\\
&+E_2(r,t+\theta-r,0)+A^0(t+\theta-r)^\top E_0(r)+Q_{10}(r,t+\theta-r)\\
&+C^0(t+\theta-r)^\top E_0(r)C_0-\big[E_4(r,0,t+\theta-r)+S_{01}(r,t+\theta-r)\\
&+B_0^\top E_1(r,t+\theta-r)^\top\big]^\top R_{00}(r)^{-1}\big[E_3(r,0)+S_{00}(r)+B_0^\top E_0(r)\big]\Big\}dr,
\end{aligned}\tag{5.16}$$

and

$$\begin{aligned}
\tilde E_2(t,\theta,\alpha)=&A_N^\top\tilde E_1(t+\theta+\delta,\alpha-\theta-\delta)^\top\mathbf{1}_{[-\delta,T-t-\delta)}(\theta)\\
&+\int_t^{(t+\theta+\delta)\wedge T}\Big\{A^0(t+\theta-r)^\top E_1(r,t+\alpha-r)^\top+E_1(r,t+\theta-r)A^0(t+\alpha-r)\\
&+C^0(t+\theta-r)^\top E_0(r)C^0(t+\alpha-r)+Q_{11}(r,t+\alpha-r,t+\theta-r)
\end{aligned}$$

$$-\big[E_4(r,0,t+\theta-r)+S_{01}(r,t+\theta-r)+B_0^\top E_1(r,t+\theta-r)^\top\big]^\top R_{00}(r)^{-1}$$
$$\times\big[E_4(r,0,t+\alpha-r)+S_{01}(r,t+\alpha-r)+B_0^\top E_1(r,t+\alpha-r)^\top\big]\Big\}dr,\alpha\geqslant\theta, \tag{5.17}$$

and for $\alpha<\theta$, $\tilde E_2(t,\theta,\alpha)=\tilde E_2(t,\alpha,\theta)^\top$. Notice that the forms of (5.10) and (5.11) are similar to (5.9). Then, the equations for $\tilde E_3(\cdot,\cdot)$, $\tilde E_4(\cdot,\cdot,\cdot)$ and $\tilde E_5(\cdot,\cdot,\cdot)$ can be constructed similarly to (5.16) and (5.17). Hence there exists a $\tau>0$ (depending only on M,l) such that $\mathscr{T}$ is a contraction mapping. By the fixed point theorem, the coupled matrix-valued Riccati equations (5.9)–(5.11) admit unique solutions.

Step 2: Let $(E_0(\cdot),E_1(\cdot,\cdot),E_2(\cdot,\cdot,\cdot),E_3(\cdot,\cdot),E_4(\cdot,\cdot,\cdot),E_5(\cdot,\cdot,\cdot))$ be the continuous solution to (5.9)–(5.11) for $T-\tau\leqslant t\leqslant T$ and $\theta,\alpha\in[-\delta,0]$. Then, $E_0(\cdot),E_1(\cdot,\cdot),E_2(\cdot,\cdot,\cdot),E_3(\cdot,\cdot)$, $E_4(\cdot,\cdot,\cdot),E_5(\cdot,\cdot,\cdot)$ satisfy Lipschitz conditions. In fact, choose $|h|$ small enough, denote

$$\begin{aligned}\mathcal{M}(t):=\sup_{\theta,\alpha\in[-\delta,0]}&\Big\{|E_1(t,\theta)-E_1(t,\theta+h)|+|E_2(t,\theta,\alpha)-E_2(t,\theta+h,\alpha)|+|E_3(t,\theta)\\&-E_3(t,\theta+h)|+|E_4(t,\theta,\alpha)-E_4(t,\theta+h,\alpha)|+|E_5(t,\theta,\alpha)-E_5(t,\theta+h,\alpha)|+|E_2(t,\theta,\alpha)\\&-E_2(t,\theta,\alpha+h)|+|E_4(t,\theta,\alpha)-E_4(t,\theta,\alpha+h)|+|E_5(t,\theta,\alpha)-E_5(t,\theta,\alpha+h)|\Big\}.\end{aligned}$$

Then, similar to (5.15)–(5.17), there exists $M'>0$ (depending only on M,τ) such that

$$\mathcal{M}(t)\leqslant M'\int_t^T\mathcal{M}(r)dr+O(h).$$

Let $h\to0$. Then, $E_0(\cdot),E_1(\cdot,\cdot),E_2(\cdot,\cdot,\cdot),E_3(\cdot,\cdot),E_4(\cdot,\cdot,\cdot),E_5(\cdot,\cdot,\cdot)$ satisfy lipschitz conditions.

Step 3: Extend the solution from $[T-\tau,T]$ to $[s,T]$. Then, (5.9)– (5.11) admit unique solutions on $[s,T]$. For example, on $[T-\tau-\tilde\tau,T-\tau]$, we substitute l with $2l$ in Step 1, where $\tilde\tau$ is the new step size. Next, we show that $E_0(\cdot),E_1(\cdot,\cdot),E_2(\cdot,\cdot,\cdot),E_3(\cdot,\cdot)$, $E_4(\cdot,\cdot,\cdot)$, $E_5(\cdot,\cdot,\cdot)$ satisfy Lipschitz conditions on $[T-\tau-\tilde\tau,T-\tau]$ in Step 2. Finally, we repeat Step 1 and Step 2 until we derive the solution on the whole interval $[s,T]$.

Remark 5.5. By the coupled matrix-valued Riccati equations (5.9)–(5.11), we obtain the closed-loop representation (5.14)—a new state feedback form. Let Problem (P) become the deterministic case, i.e. the diffusion term disappears in (5.7). Then, (5.9)– (5.11) are similar to (2.33)–(2.38) in [12]. Moreover, Theorem 5.3 is derived similarly, when the coefficients of the state equation (5.7) are time-variant.

6. Closed-loop solvability. In this section, we study a stochastic optimal control problem which involves only state delay not control delay. The general case is open, due to some technical reasons, up to now. By an equivalent transformed control problem, we define the closed-loop solvability for the original delayed control problem, and assure it by the solvability of a differential operator-valued Riccati equation.

First we reformulate the optimal control problem as follows. Now the state equation (2.3) becomes the following SDDE:

$$\begin{cases}dX(t)=\Big[\int_{[-\delta,0]}A(d\theta)X_t(\theta)+B_0u(t)\Big]dt+\Big[\int_{[-\delta,0]}C(d\theta)X_t(\theta)+D_0u(t)\Big]dW(t), t\in[s,T],\\X(s)=x,\quad X(t)=\varphi(t-s),\ t\in[s-\delta,s),\end{cases}\tag{6.1}$$

where $\int_{[-\delta,0]}A(d\theta)\tilde\varphi(\theta)$ and $\int_{[-\delta,0]}C(d\theta)\tilde\varphi(\theta)$ are defined by (2.1) and (2.2), for any $\tilde\varphi\in\mathfrak{L}$. The cost functional (2.4) becomes:

$$\begin{aligned}J(s,x,\varphi(\cdot);u(\cdot))=\mathbb{E}\int_s^T&\Big[\big\langle Q_{00}(t)X(t),X(t)\big\rangle+2\int_{-\delta}^0\big\langle Q_{10}(t,\theta)^\top X(t+\theta),X(t)\big\rangle d\theta\\&+\int_{[-\delta,0]^2}\big\langle Q_{11}(t,\theta,\theta')X(t+\theta),X(t+\theta')\big\rangle d\theta'd\theta+2\big\langle S_{00}(t)X(t),u(t)\big\rangle\\&+2\int_{-\delta}^0\big\langle S_{01}(t,\theta)X(t+\theta),u(t)\big\rangle d\theta+\big\langle R_{00}(t)u(t),u(t)\big\rangle\Big]dt+\mathbb{E}\Big[\big\langle G_{00}X(T),X(T)\big\rangle\end{aligned}$$

$$+2\int_{-\delta}^{0}\big\langle G_{10}(\theta)^\top X(T+\theta),X(T)\big\rangle d\theta+\int_{[-\delta,0]^2}\big\langle G_{11}(\theta,\theta')X(T+\theta),X(T+\theta')\big\rangle d\theta' d\theta\Big]. \tag{6.2}$$

We restate the control problem studied in this section as follows.

Problem ($\tilde{\mathbf{P}}$). For any $(s,x,\varphi)\in[0,T)\times\mathfrak{M}$, to find a $\bar u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (6.1) is satisfied and

$$J(s,x,\varphi(\cdot);\bar u(\cdot))=\inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)}J(s,x,\varphi(\cdot);u(\cdot)):=V(s,x,\varphi(\cdot)).$$

As in Section 3, we transform the delayed state equation (6.1) in $\mathbb{R}^n$ into one in $\mathfrak{M}$ without delay. Now the transformed state equation (3.10) becomes

$$\mathbf{X}(t)=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde B u(r)dr+\int_s^t\Phi(t-r)\Big(\tilde C\mathbf{X}(r)+\tilde D u(r)\Big)dW(r),\ t\in[s,T], \tag{6.3}$$

where $\xi:=\begin{pmatrix}x\\ \varphi\end{pmatrix}$, $\Phi(\cdot),\tilde C$ are defined as (3.1) and (3.3), $\tilde B,\tilde D$ are redefined as $\tilde B:\mathbb{R}^m\to\mathfrak{M}$, $u\mapsto\begin{pmatrix}B_0u\\ 0\end{pmatrix}$, and $\tilde D:\mathbb{R}^m\to\mathfrak{M}$, $u\mapsto\begin{pmatrix}D_0u\\ 0\end{pmatrix}$, for any $u\in\mathbb{R}^m$. The cost (3.11) becomes

$$\begin{aligned}J(s,\xi;u(\cdot))&=J(s,x,\varphi(\cdot);u(\cdot))=\mathbb{E}\Big\{\int_s^T\Big[\big\langle\tilde Q(t)\mathbf{X}(t),\mathbf{X}(t)\big\rangle_{\mathfrak{M}}\\&+2\big\langle\tilde S_0(t)\mathbf{X}(t),u(t)\big\rangle+\big\langle\tilde R_{00}(t)u(t),u(t)\big\rangle\Big]dt+\big\langle\tilde G\mathbf{X}(T),\mathbf{X}(T)\big\rangle_{\mathfrak{M}}\Big\}.\end{aligned} \tag{6.4}$$

Then we restate Problem (EP), and define the closed-loop solvability for Problem ($\tilde{\mathrm{P}}$).

Problem ($\widetilde{\mathbf{EP}}$). For any $(s,\xi)\in[0,T)\times\mathfrak{M}$, to find a $\bar u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (6.3) is satisfied and

$$J(s,\xi;\bar u(\cdot))=\inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)}J(s,\xi;u(\cdot)):=V(s,\xi).$$

DEFINITION 6.1. *Any* $K(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ *is called a* closed-loop strategy of Problem ($\tilde{\mathrm{P}}$) *on* $[s,T]$. *For any* $K(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ *and* $(x,\varphi)\in\mathfrak{M}$, *let* $\xi\equiv\begin{pmatrix}x\\ \varphi\end{pmatrix}$, $\mathbf{X}(\cdot)\equiv\mathbf{X}(\cdot\,;s,\xi,K(\cdot))$ *be the solution to the following equation:*

$$\mathbf{X}(t)=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde BK(r)\mathbf{X}(r)dr+\int_s^t\Phi(t-r)\Big[\tilde C\mathbf{X}(r)+\tilde DK(r)\mathbf{X}(r)\Big]dW(r), \tag{6.5}$$

and

$$u(t)=K(t)\mathbf{X}(t),\quad t\in[s,T].$$

Then, $(\mathbf{X}(\cdot),u(\cdot))$ *is called the* outcome pair *of* $K(\cdot)$ *on* $[s,T]$ *corresponding to the initial trajectory* (x,φ); $\mathbf{X}(\cdot)$, $u(\cdot)$ *are called the corresponding* closed-loop state *and* closed-loop outcome control, *respectively.*

DEFINITION 6.2. *A closed-loop strategy* $\bar K(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ *is said to be* optimal *on* $[s,T]$ *if*

$$J\big(s,\xi;\bar K(\cdot)\bar{\mathbf{X}}(\cdot)\big)\le\ J\big(s,\xi;u(\cdot)\big),\quad\forall u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m),\ \forall\xi=\begin{pmatrix}x\\ \varphi\end{pmatrix}\in\mathfrak{M},$$

where $\bar{\mathbf{X}}(\cdot)$ *is the closed-loop state corresponding to* $(\bar K(\cdot),x,\varphi)$. *If there (uniquely) exists an optimal closed-loop strategy on* $[s,T]$, Problem ($\tilde{\mathrm{P}}$) *is said to be* (uniquely) closed-loop solvable on $[s,T]$.

Introduce the following linear operator-valued equation:

$$\begin{cases}\dot P(t)+P(t)(\tilde A+\tilde B\bar K(t))+(\tilde A+\tilde B\bar K(t))^*P(t)+(\tilde C+\tilde D\bar K(t))^*P(t)(\tilde C+\tilde D\bar K(t))\\ \quad+\tilde Q(t)+\bar K(t)^*\tilde R_{00}(t)\bar K(t)+\bar K(t)^*\tilde S_0(t)+\tilde S_0(t)^*\bar K(t)=0,\ t\in[s,T],\\ P(T)=\tilde G.\end{cases} \tag{6.6}$$

Then, we explore the necessary conditions of closed-loop solvability for Problem ($\tilde{\mathrm{P}}$).

THEOREM 6.3. *Let* (A1)–(A2) *hold. Suppose* $\bar K(\cdot)$ *is the optimal closed-loop strategy of* Problem $(\tilde{\mathrm P})$ *on* $[s,T]$. *Then,*

$$\tilde R_{00}(t)+\tilde D^*P(t)\tilde D\geq 0,\quad \text{a.e.},$$

$$\big[\tilde R_{00}(t)+\tilde D^*P(t)\tilde D\big]\bar K(t)+\tilde B^*P(t)+\tilde D^*P(t)\tilde C+\tilde S_0(t)=0,\quad \text{a.e.}, \tag{6.7}$$

where $P(\cdot)$ *satisfies (6.6).*

Proof. For any $v(\cdot)\in L^2_{\mathbb F}(s,T;\mathbb R^m)$ and $t\in[s,T]$, consider the following SEE:

$$\begin{cases} dz(t)=\big[\tilde Az(t)+\tilde B\bar K(t)z(t)+\tilde Bv(t)\big]dt+\big[\tilde Cz(t)+\tilde D\bar K(t)z(t)+\tilde Dv(t)\big]dW(t),\\ z(s)=\xi,\end{cases} \tag{6.8}$$

where $\tilde A$ is defined as (3.2). Then, applying Itô formula to $\langle P(\cdot)z(\cdot),z(\cdot)\rangle$ (substituting $\tilde A$ with its Yosida approximation $\tilde A_\lambda$, and letting $\lambda\to\infty$), we obtain

$$\begin{aligned} J(s,\xi;\bar K(\cdot)z(\cdot)+v(\cdot))&=\mathbb E\langle P(s)\xi,\xi\rangle+\mathbb E\int_s^T\big[\big\langle\big(\tilde R_{00}(t)+\tilde D^*P(t)\tilde D\big)v(t),v(t)\big\rangle\\ &\quad+2\big\langle\big(\tilde B^*P(t)+\tilde D^*P(t)\tilde C+\tilde R_{00}(t)\bar K(t)+\tilde S_0(t)+\tilde D^*P(t)\tilde D\bar K(t)\big)z(t),v(t)\big\rangle\big]dt.\end{aligned}$$

Since $\bar K(\cdot)$ is the optimal closed-loop strategy, we have

$$\begin{aligned}&\mathbb E\int_s^T\Big[2\Big\langle\Big(\tilde B^*P(t)+\tilde D^*P(t)\tilde C+\tilde R_{00}(t)\bar K(t)+\tilde S_0(t)+\tilde D^*P(t)\tilde D\bar K(t)\Big)z(t),v(t)\Big\rangle\\ &\qquad+\Big\langle\Big(\tilde R_{00}(t)+\tilde D^*P(t)\tilde D\Big)v(t),v(t)\Big\rangle\Big]dt\geqslant 0,\ \forall v(\cdot)\in L^2_{\mathbb F}(s,T;\mathbb R^m).\end{aligned} \tag{6.9}$$

In the following, we aim to prove that

$$\tilde R_{00}(t)+\tilde D^*P(t)\tilde D\geqslant 0,\ \text{a.e.} \tag{6.10}$$

Suppose there exists $\Omega_0\subseteq[s,T]$ and $|\Omega_0|>\frac1l$, for some $l>0$, such that $\tilde R_{00}(t)+\tilde D^*P(t)\tilde D<0$ on Ω_0. Without loss of generality, assume that there exists $\beta>0$ such that $\tilde R_{00}(t)+\tilde D^*P(t)\tilde D\leqslant-\beta I$. Then, we can choose a sequence of Borel measurable sets $\{\Omega_k\}$ such that $\Omega_k\subseteq\Omega_0$ and $|\Omega_k|=\frac{1}{l+k}$. Let $\xi=0$, $v_k=(\sqrt k,0,\cdots,0)^\top I_{\Omega_k}(t)$, and $z_k(\cdot)$ be the corresponding solution to (6.8). Then, we have

$$\sup_{s\leqslant t\leqslant T}\mathbb E|z_k(t)|^2\leqslant M\mathbb E\int_s^T|v_k(t)|^2dt=\frac{k}{k+l}M\leqslant M,$$

here and after, M is a generic constant. By (6.9), we have

$$\begin{aligned}0&\leqslant\varlimsup_{k\to\infty}\mathbb E\int_s^T\big\langle\big(\tilde R_{00}(t)+\tilde D^*P(t)\tilde D\big)v_k(t),v_k(t)\big\rangle dt+2\varlimsup_{k\to\infty}\mathbb E\int_s^T\big\langle\big(\tilde B^*P(t)\\ &\quad+\tilde D^*P(t)\tilde C+\tilde R_{00}(t)\bar K(t)+\tilde S_0(t)+\tilde D^*P(t)\tilde D\bar K(t)\big)z_k(t),v_k(t)\big\rangle dt\\ &\leqslant-\beta\frac{k}{k+l}+M\sqrt{\frac{k}{k+l}}\Big(\int_{\Omega_k}||\bar K(t)||^2_{\mathscr L(\mathfrak M,\mathbb R^m)}dt\Big)^{\frac12}\to-\beta,\ \text{as }k\to\infty,\end{aligned}$$

which is a contradiction! Thus, (6.10) holds. It remains to prove the second equality in (6.7). $\bar K(\cdot)$ is the optimal closed-loop strategy of Problem $(\tilde{\mathrm P})$ on $[s,T]$, thus is also optimal on $[r,T]$ for any $r\in(s,T]$, then (6.9) holds for any $r\in(s,T]$. Choose $\xi\in\mathfrak M$, $v_j(t)=\frac1jv(t)$, $v(\cdot)\in L^2_{\mathbb F}(s,T;\mathbb R^m)$, let $z_j(\cdot)$ be the solution to the following SEE:

$$\begin{cases}dz_j(t)=[\tilde Az_j(t)+\tilde B\bar K(t)z_j(t)+\tilde Bv_j(t)]dt+[\tilde Cz_j(t)+\tilde D\bar K(t)z_j(t)+\tilde Dv_j(t)]dW(t),\ t\in[r,T],\\ z_j(r)=\xi.\end{cases}$$

Then, by (6.9), $\forall v(\cdot)\in L^2_{\mathbb F}(s,T;\mathbb R^m)$, we derive

$$\varlimsup_{j\to\infty}\mathbb E\int_r^T\big\langle\big(\tilde B^*P(t)+\tilde D^*P(t)\tilde C+\tilde R_{00}(t)\bar K(t)+\tilde S_0(t)+\tilde D^*P(t)\tilde D\bar K(t)\big)z_j(t),v(t)\big\rangle dt\geqslant 0. \tag{6.11}$$

Consider the following SEE:

$$\begin{cases}d\tilde z(t)=\Big(\tilde A\tilde z(t)+\tilde B\bar K(t)\tilde z(t)\Big)dt+\Big(\tilde C\tilde z(t)+\tilde D\bar K(t)\tilde z(t)\Big)dW(t),\quad t\in[r,T],\\ \tilde z(r)=\xi.\end{cases}$$

Then, we have

$$\sup_{r\leqslant t\leqslant T}\mathbb E|z_j(t)-\tilde z(t)|^2\leqslant\mathbb E\int_r^T|v_j(t)|^2dt\to0,\ \text{as }j\to\infty,$$

which and (6.11) imply that

$$\mathbb{E}\int_r^T\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{R}_{00}(t)\bar{K}(t)+\tilde{S}_0(t)+\tilde{D}^*P(t)\tilde{D}\bar{K}(t)\Big)\tilde{z}(t),v(t)\Big\rangle dt\geqslant 0,$$

for any $v(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$. Choose $v(t)=v\mathbf{1}_{[r,r+\varepsilon]}(t)$, $v\in\mathbb{R}^m$. Then, we deduce

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_r^{r+\varepsilon}\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{R}_{00}(t)\bar{K}(t)+\tilde{S}_0(t)+\tilde{D}^*P(t)\tilde{D}\bar{K}(t)\Big)\xi,v\Big\rangle dt=0,$$

for any $\xi\in\mathfrak{M}$, $v\in\mathbb{R}^m$. By the arbitrariness of ξ and v, the second equality of (6.7) holds. Hence we complete the proof. □

Next we give the sufficient conditions of the closed-loop solvability for Problem $(\tilde{\mathrm{P}})$.

THEOREM 6.4. *Let* (A1)–(A2) *hold. Suppose* $\tilde{R}_{00}+\tilde{D}^*P\tilde{D}\geq 0$, $\mathcal{R}(\tilde{B}^*P+\tilde{D}^*P\tilde{C}+\tilde{S}_0)\subseteq\mathcal{R}(\tilde{R}_{00}+\tilde{D}^*P\tilde{D})$, *with* $P(\cdot)$ *satisfying the Riccati equation* (6.6). *Here*

$$\begin{aligned}\bar{K}(t)=&-\Big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\Big)^{\dagger}\Big[\tilde{B}^*P(t)+\tilde{S}_0(t)+\tilde{D}^*P(t)\tilde{C}\Big]\\ &+\Big[I-\Big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\Big)^{\dagger}\Big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\Big)\Big]\theta(t),\ \text{a.e.},\end{aligned}\tag{6.12}$$

for any $\theta(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$. *Suppose* $\bar{K}(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$. *Then, it is the optimal closed-loop strategy of* Problem $(\tilde{\mathrm{P}})$, *and the value function is as follows*

$$V(s,\xi)=\langle P(s)\xi,\xi\rangle_{\mathfrak{M}}.\tag{6.13}$$

Proof. Denote $\mathcal{M}(t):=\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}$. Then, by (6.12) and $\mathcal{R}(\tilde{B}^*P+\tilde{D}^*P\tilde{C}+\tilde{S}_0)\subseteq\mathcal{R}(\mathcal{M})$, we derive

$$\mathcal{M}(t)\bar{K}(t)+\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)=0,\quad\text{a.e..}\tag{6.14}$$

By (6.3) and applying Itô formula to $\langle P(\cdot)\mathbf{X}(\cdot),\mathbf{X}(\cdot)\rangle$, we obtain

$$\begin{aligned}J(s,\xi;u(\cdot))=\mathbb{E}\Big\{&\langle P(s)\xi,\xi\rangle+\int_s^T\Big[\langle\mathcal{M}(t)u(t),u(t)\rangle-\langle\mathcal{M}(t)\bar{K}(t)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\rangle-2\Big\langle\Big(\tilde{B}^*P(t)\\ &+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)\Big)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\Big\rangle+2\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)\Big)\mathbf{X}(t),u(t)\Big\rangle\Big]dt\Big\}.\end{aligned}$$

Noting (6.14), we have

$$\begin{aligned}&-\langle\mathcal{M}(t)\bar{K}(t)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\rangle-2\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)\Big)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\Big\rangle\\ &=-\langle\mathcal{M}(t)\bar{K}(t)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\rangle+2\langle\mathcal{M}(t)\bar{K}(t)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\rangle=\langle\mathcal{M}(t)\bar{K}(t)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\rangle,\end{aligned}$$

which yields

$$J(s,\xi;u(\cdot))=\mathbb{E}\Big\{\langle P(s)\xi,\xi\rangle+\int_s^T\langle\mathcal{M}(t)\big(u(t)-\bar{K}(t)\mathbf{X}(t)\big),u(t)-\bar{K}(t)\mathbf{X}(t)\rangle dt\Big\}.$$

Thus, we complete the proof. □

We summarize the above discussion and characterize the closed-loop solvability for Problem $(\tilde{\mathrm{P}})$.

THEOREM 6.5. *Let* (A1)–(A2) *hold. Then,* $\bar{K}(\cdot)$ *is the optimal closed-loop strategy of* Problem $(\tilde{\mathrm{P}})$ *on* $[s,T]$ *if and only if*

(i) $\bar{K}(\cdot)$ *is given by (6.12), where* $P(\cdot)$ *satisfies the differential operator-valued Riccati equation (6.6),*

(ii) $\tilde{R}_{00}+\tilde{D}^*P\tilde{D}\geq 0$, $\mathcal{R}(\tilde{B}^*P+\tilde{D}^*P\tilde{C}+\tilde{S}_0)\subseteq\mathcal{R}(\tilde{R}_{00}+\tilde{D}^*P\tilde{D})$,

(iii) $\bar{K}(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$.

In the case, the value function is given by (6.13).

Remark 6.6. In Theorem 6.5, we give some sufficient conditions for the solvability of the Riccati equation (6.6). Moreover, we overcome the difficulties of decoupling forward delayed state equations and backward advanced adjoint equations, by introducing the closed-loop strategy and the auxiliary equation (6.6). When B_0,C_0,D_0,$C^0(\theta)$ depend on t, Theorem 6.5 is derived similarly.

Inspired by (6.6), recall that $\hat{\delta}(\cdot)$ is the delta function, denote $\mathfrak{R}(t) := R_{00}(t) + D_0^\top \mathcal{E}_0(t)D_0$, and for almost everywhere $t \in [s,T]$, $\theta,\alpha \in [-\delta,0]$, introduce the following coupled matrix-valued Riccati equation:

$$
(6.15)\quad
\begin{cases}
\dot{\mathcal{E}}_0(t)+A_0^\top\mathcal{E}_0(t)+\mathcal{E}_0(t)A_0+\mathcal{E}_1(t,0)+\mathcal{E}_1(t,0)^\top+C_0^\top\mathcal{E}_0(t)C_0+Q_{00}(t)\\
\quad-\Big[S_{00}(t)+B_0^\top\mathcal{E}_0(t)+D_0^\top\mathcal{E}_0(t)C_0\Big]^\top\mathfrak{R}(t)^\dagger\Big[S_{00}(t)+B_0^\top\mathcal{E}_0(t)+D_0^\top\mathcal{E}_0(t)C_0\Big]=0,\\
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}\Big)\mathcal{E}_1(t,\theta)+A_0^\top\mathcal{E}_1(t,\theta)+\mathcal{E}_2(t,0,\theta)+\mathcal{E}_0(t)\Big[\sum_{i=1}^{N-1}A_i\hat{\delta}(\theta-\theta_i)+A^0(\theta)\Big]\\
\quad+Q_{10}(t,\theta)^\top+C_0^\top\mathcal{E}_0(t)C^0(\theta)-\Big[S_{00}(t)+B_0^\top\mathcal{E}_0(t)+D_0^\top\mathcal{E}_0(t)C_0\Big]^\top\mathfrak{R}(t)^\dagger\\
\quad\times\Big[S_{01}(t,\theta)+B_0^\top\mathcal{E}_1(t,\theta)+D_0^\top\mathcal{E}_0(t)C^0(\theta)\Big]=0,\\
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}-\dfrac{\partial}{\partial\alpha}\Big)\mathcal{E}_2(t,\theta,\alpha)+\Big[\sum_{i=1}^{N-1}A_i\hat{\delta}(\theta-\theta_i)+A^0(\theta)\Big]^\top\mathcal{E}_1(t,\alpha)+\mathcal{E}_1(t,\theta)^\top\Big[A^0(\alpha)\\
\quad+\sum_{i=1}^{N-1}A_i\hat{\delta}(\alpha-\theta_i)\Big]+C^0(\theta)^\top\mathcal{E}_0(t)C^0(\alpha)+Q_{11}(t,\alpha,\theta)-\Big[S_{01}(t,\theta)+B_0^\top\mathcal{E}_1(t,\theta)\\
\quad+D_0^\top\mathcal{E}_0(t)C^0(\theta)\Big]^\top\mathfrak{R}(t)^\dagger\Big[S_{01}(t,\alpha)+B_0^\top\mathcal{E}_1(t,\alpha)+D_0^\top\mathcal{E}_0(t)C^0(\alpha)\Big]=0,\\
\mathcal{E}_0(T)=G_{00},\quad \mathcal{E}_1(T,\theta)=G_{10}(\theta)^\top,\quad \mathcal{E}_1(t,-\delta)=\mathcal{E}_0(t)A_N,\\
\mathcal{E}_2(T,\theta,\alpha)=G_{11}(\alpha,\theta),\ \mathcal{E}_2(t,-\delta,\alpha)=A_N^\top\mathcal{E}_1(t,\alpha),\ \mathcal{E}_2(t,\theta,-\delta)=\mathcal{E}_1(t,\theta)^\top A_N.
\end{cases}
$$

Then, we go back to the original delayed control problem $(\tilde{\mathrm{P}})$, and give a clear characterization of its closed-loop solvability.

Theorem 6.7. *Suppose all coefficients of Problem* $(\tilde{\mathrm{P}})$ *are continuous and* $\mathfrak{R}\geq 0$. *Let* $\mathcal{E}_0(t)$, $\mathcal{E}_1(t,\theta)$, $\mathcal{E}_2(t,\theta,\alpha)$, $t\in[s,T]$, $\theta,\alpha\in[-\delta,0]$, *be continuous functions satisfying the equation* (6.15), *and* $\mathcal{E}_0(t)=\mathcal{E}_0(t)^\top$, $\mathcal{E}_2(t,\theta,\alpha)=\mathcal{E}_2(t,\alpha,\theta)^\top$. *Moreover,*

$$
(6.16)\quad \big(B_0^\top\mathcal{E}_0(t)+S_{00}(t)+D_0^\top\mathcal{E}_0(t)C_0\big)x+\int_{-\delta}^0\Big(B_0^\top\mathcal{E}_1(t,\theta)+D_0^\top\mathcal{E}_0(t)C^0(\theta)+S_{01}(t,\theta)\Big)\varphi(\theta)d\theta\in\mathcal{R}(\mathfrak{R}(t)),\ \forall x\in\mathbb{R}^n,\ \varphi\in\mathfrak{L}.
$$

Let $\bar{K}(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ *be given by*

$$
(6.17)\quad \bar{K}(t)\xi=-\mathfrak{R}(t)^\dagger\Big[\Big(B_0^\top\mathcal{E}_0(t)+S_{00}(t)+D_0^\top\mathcal{E}_0(t)C_0\Big)x+\int_{-\delta}^0\Big(B_0^\top\mathcal{E}_1(t,\theta)+D_0^\top\mathcal{E}_0(t)C^0(\theta)+S_{01}(t,\theta)\Big)\varphi(\theta)d\theta\Big]+\big[I-\mathfrak{R}(t)^\dagger\mathfrak{R}(t)\big]\theta(t)\xi,\ \ \theta(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m)),\ \forall\xi=\begin{pmatrix}x\\ \varphi\end{pmatrix}.
$$

Then, $\bar{K}(\cdot)$ *is the optimal closed-loop strategy for* Problem $(\tilde{\mathrm{P}})$*, and the value function is as follows:*

$$
V(s,x,\varphi(\cdot))=\big\langle\mathcal{E}_0(s)x,x\big\rangle+2\int_{-\delta}^0\big\langle\mathcal{E}_1(s,\theta)\varphi(\theta),x\big\rangle d\theta+\int_{[-\delta,0]^2}\big\langle\mathcal{E}_2(s,\theta,\alpha)\varphi(\alpha),\varphi(\theta)\big\rangle d\alpha d\theta.
$$

Proof. For any $u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, let $X(\cdot)$ be the state satisfying (6.1). Define

$$
\Gamma(t):=\langle\mathcal{E}_0(t)X(t),X(t)\rangle+2\int_{-\delta}^0\langle\mathcal{E}_1(t,\theta)X(t+\theta),X(t)\rangle d\theta+\int_{[-\delta,0]^2}\langle\mathcal{E}_2(t,\theta,\alpha)X(t+\alpha),X(t+\theta)\rangle d\alpha d\theta.
$$

Then, by (6.15)–(6.17), with some computations we derive

$$
\begin{aligned}
&d\Gamma(t)+\big\langle Q_{00}(t)X(t),X(t)\big\rangle+2\int_{-\delta}^0\big\langle Q_{10}(t,\theta)^\top X(t+\theta),X(t)\big\rangle d\theta+\int_{[-\delta,0]^2}\big\langle Q_{11}(t,\theta,\theta')X(t+\theta),\\
&X(t+\theta')\big\rangle d\theta' d\theta+2\big\langle S_{00}(t)X(t),u(t)\big\rangle+2\int_{-\delta}^0\big\langle S_{01}(t,\theta)X(t+\theta),u(t)\big\rangle d\theta+\big\langle R_{00}(t)u(t),u(t)\big\rangle\\
&=\Big\langle\mathfrak{R}(t)\Big[u(t)-\bar{K}(t)\begin{pmatrix}X(t)\\ X_t\end{pmatrix}\Big],u(t)-\bar{K}(t)\begin{pmatrix}X(t)\\ X_t\end{pmatrix}\Big\rangle,\quad \text{a.e. } t\in[s,T].
\end{aligned}
$$

Integrating both sides of which from s to T, we complete the proof. ∎

COROLLARY 6.8. *Suppose all coefficients of Problem* $(\tilde{\mathrm{P}})$ *are continuous. Let* $\mathcal{E}_0(t)$, $\mathcal{E}_1(t,\theta)$, $\mathcal{E}_2(t,\theta,\alpha)$, $t\in[s,T]$, $\theta,\alpha\in[-\delta,0]$, *be continuous functions satisfying the coupled matrix-valued Riccati equation* (6.15), *and* $\mathfrak{R}(t)=R_{00}(t)+D_0^\top\mathcal{E}_0(t)D_0>0$. *Let continuous functions* $E_0(t)$, $E_1(t,\theta)$, $E_2(t,\theta,\alpha)$, $E_3(t,\theta)$, $E_4(t,\theta,\alpha)$, $E_5(t,\theta,\alpha)$, $t\in[s,T]$, $\theta,\alpha\in[-\delta,0]$, *satisfy the coupled matrix-valued Riccati equations* (5.9)–(5.11). *Then,* $\mathcal{E}_0(t)=E_0(t), \mathcal{E}_1(t,\theta)=E_1(t,\theta)^\top, \mathcal{E}_2(t,\theta,\alpha)=E_2(t,\theta,\alpha), E_3(t,\theta,\alpha), E_4(t,\theta,\alpha), E_5(t,\theta,\alpha)=0$; *and the closed-loop outcome control of Problem* $(\tilde{\mathrm{P}})$ *is as follows:*

$$\bar{u}(t)=\bar{K}(t)\bar{\mathbf{X}}(t), \tag{6.18}$$

where $\bar{K}(\cdot)$ *is defined by* (6.17) *and* $\bar{\mathbf{X}}(\cdot)$ *is the solution to* (6.5). *In this case,* (6.18) *is the same as the closed-loop representation of the open-loop optimal control* (5.13).

Remark 6.9. Similar to Remark 5.4, let $A_i,D_0,G_{00},G_{10},G_{11}=0, i=1,\cdots,N-1$. Then, (6.15) admits a unique solution. Theorem 6.5 assures the closed-loop solvability for Problem $(\tilde{\mathrm{P}})$ by the solvability of the differential operator-valued Riccati equation (6.6). Furthermore, by the coupled matrix-valued Riccati equation (6.15), Theorem 6.7 explicitly represents the optimal closed-loop strategy $\bar{K}(\cdot)$ using the coefficients of the original delayed control systems. When delay disappears in Problem $(\tilde{\mathrm{P}})$, Theorem 6.7 is similar to the sufficient part of Theorem 2.4.3 in [31]. When the coefficients of the state equation (6.1) are time-variant, Theorem 6.7 also holds.

7. Concluding remarks. This paper studies the linear quadratic optimal control problem for a delayed stochastic system with both state delay and control delay in the diffusion term. We transform it into an infinite dimensional problem without delay, ensuring the open-loop solvability through a constrained forward-backward stochastic evolution system and a convexity condition. We also provide a closed-loop representation using a coupled matrix-valued Riccati equation and assure the closed-loop solvability via a differential operator-valued Riccati equation, ultimately clarifying the original delayed optimal control problem.

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